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TEXTS AND READINGS IN MATHEMATICS 31

Lectures on Curves on an Algebraic Surface

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with a section by
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Lectures on Curves on an Algebraic Surface

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DEDICATION

The contributors to this volume dedicate their work to the memory of

M. K. Fort, Jr.

whose warmth and good will have been felt by the entire mathematical community.

INTRODUCTION

These notes are being printed in exactly the form in which they were first written and distributed: as class notes, supplementing and working out my oral lectures. As such, they are far from polished and ask a lot of the reader. In the words of the ex-editor of a well-known journal they are written in a style "seldom seen except in personal letters between close friends." Be that as it may, my hope is that a well-intentioned reader will still be able to penetrate these notes and learn something of the beautiful geometry on an algebraic surface.

It was expected, when these notes were written, that the reader had the following background: he had taken a graduate course in commutative algebra, he had studied some Algebraic Geometry and, in particular, he had some acquaintance with the theory of curves, and the theory of schemes, and of their cohomology (e.g., Dieudonne's Maryland and Montreal Lecture Notes). Nonetheless, both to fix ideas, and to prove some specialized results that are needed later, Lectures 3-10 are devoted to a quick and rather breezy digression into the general theory of schemes. Lecture 11 summarizes what we need from the theory of curves. I apologize to any reader who, hoping that he would find here in these 60 odd pages an easy and concise introduction to schemes, instead became hopelessly lost in a maze of unproven assertions and undeveloped suggestions. _From Lecture 12 on, we have proven everything that we need._

The goal of these lectures is a complete clarification of one "theorem" on Algebraic surfaces: the so-called completeness of the characteristic linear system of a good complete algebraic system of curves, on a surface F. If the characteristic is 0, this theorem was first proven by Poincaré (cf. References) in 1910 by analytic methods. Until about 1960, no algebraic proof of this purely algebraic theorem was known.* In 1955, Igusa had shown that the theorem, as stated, was false in characteristic p thus making the theorem appear even more analytic in nature. But about 1960, a truly amazing development occurred: in the course of working out the master plan that he had laid out for Algebraic Geometry—incorporating some of the key ideas of Kodaira's and Spencer's deformation theory—Grothendieck had occasion to write out some of the Corollaries of his theory (cf. _his Bourbaki exposé_ 221, pp. 23-24). Putting his results together with a

* Although an endless and depressing controversy obscured this fact.

result of Cartier—that group schemes in characteristic 0 are reduced—one finds that this old problem has been completely solved: a) a purely algebraic proof is available in characteristic 0, b) all the machinery is ready at hand for obtaining, in characteristic p, necessary and sufficient conditions for the validity of the theorem. What was the key, the essential point which the Italians had overlooked? There is no doubt at all that it is the systematic use of nilpotent elements: in particular, a systematic analysis of all families of curves on a surface over a parameter space with only one point, but with non-trivial nilpotent structure sheaf. The Italians had, in a sense, done this, but only when the ring of functions on the base was Study's ring of dual numbers $k[\varepsilon]/(\varepsilon^2)$. This is the same as looking at first-order deformations of a curve. But they ignored higher order nilpotents and higher order deformations.

The outline of these lectures is as follows—Lectures 1 and 2 give an intuitive introduction to the problem and present in outline 2 analytic proofs. Lectures 3 through 10 recall basic notions about schemes. Lectures 11 through 21 deal with basic questions on the theory of surfaces. In particular, they give a construction of universal families of curves on a surface—the so-called Hilbert scheme; and of universal families of divisor classes on a surface—the so-called Picard scheme. Lectures 22 through 27 deal with the application of the whole theory to the main problem: these include a long lecture by G. Bergman giving a self-contained description of the Witt ring schemes.

I would like to call attention to several generalizations and applications of our results which were omitted so as to get directly to the main result.

a) The method by which we have constructed the universal family of curves on a surface F gives without any change a construction of the universal flat family of subschemes of any scheme X, projective over a noetherian S, i.e., of the Hilbert scheme. In particular, the explicit estimates obtained in Lecture 14 enable ont to carry through this construction—which is Grothendieck's original construction—without the indirect arguments using the concept of "limited families" which he used (cf. his "Fondements").

b) The method by which we have constructed the Picard scheme of a surface F generalizes so as to construct the Picard scheme of any scheme X, projective and flat over a noetherian S, whose geometric fibres over S are reduced and connected and such that the components of its actual fibres over S are absolutely irreducible. This construction is related to the one I outlined at the International Congress of 1962, and ties up with the methods used in Chapters 3 and 7 of my book Geometric Invariant Theory.

c) One can use the results of Lecture 18 to give a very easy proof of the Riemann Hypothesis for curves over finite fields. This is the proof of Mattuck-Tate (cf. References). If you have read through Lecture 18, and know the formulation of the Riemann Hypothesis via the Frobenius morphism, you can read their paper without difficulty and you should.

Cambridge
March, 1966

CONTENTS

LECTURE 1

RAW MATERIAL ON CURVES ON SURFACES, AND THE PROBLEMS SUGGESTED

We shall be concerned entirely with algebraic geometry over a fixed algebraically closed field k (of arbitrary characteristic). Our chief purpose is to study the geometry on a non-singular algebraic surface F, projective over k, and, in particular, the families of curves C on F.

By a curve we mean either a finite sum of irreducible, 1-dimensional subvarieties of F, with positive multiplicity: $\Sigma n_i C_i$, or a sheaf of principal ideals on F. [These are equivalent concepts—for precise definitions, cf. Lecture 9.]

Example 1: $F = P_2$. Then, as is well-known, every curve C on P_2 is defined by a homogeneous form $F(x_0, x_1, x_2)$. In particular, one can attach to C its degree d, i.e., the degree of F, and the family of all curves of degree d is parametrized by the set of all F of degree d, up to scalars: i.e., by a projective space of dimension

$$\frac{(d+1)(d+2)}{2} - 1$$

Example 2: $F = P_1 \times P_1$ (i.e., a quadric in P_3). Then every curve C on F is defined by a bi-homogeneous form

$$F(x_0, x_1; y_0, y_1)$$

with two degrees d and e. d and e can be interpreted as the degrees of the coverings

$$p_1, p_2: C \to P_1$$

given by the two projections of $P_1 \times P_1$ onto P_1. Again, for every d and e, there is a single family of curves parametrized by a projective space, this time of dimension;

$$(d+1)(e+1) - 1$$

The phenomenon of the last two examples can be generalized by the concept of a linear system. If f is an algebraic function on F, let, as usual, (f) stand for the formal sum:

$$\sum_{\substack{\text{all 1-dimensional}\\ \text{irreducible subvarieties}\\ E}} \mathrm{ord}_E(f) \cdot E$$

where $\mathrm{ord}_E(f)$ is the order of the zero or pole of f at E. Then associated to any curve C one has the vector space of functions with poles only at C:

$$\mathfrak{L}(C) = \{f \mid (f) + C \geq 0\}$$

(Here $\Sigma n_i E_i \geq 0$ means all $n_i \geq 0$.) If $f_0, \ldots, f_n$ are a basis of $\mathfrak{L}(C)$, one then can define the following family of curves, which contains C:

$$C_\alpha = (\Sigma \alpha_i f_i) + C$$

Since C_α only depends on the ratios of the α_i, this is an irreducible family of curves parametrized by a projective space of dimension:

$$\dim \mathfrak{L}(C) - 1$$

Linear systems are the simplest families of curves on a surface F and the only type occurring in Examples 1 and 2.

Definition: Two curves C_1 and C_2 are linearly equivalent if equivalently:

i) $\exists$ a function f on F such that $(f) = C_1 - C_2$, or

ii) C_1, C_2 are in the same linear system.

We write $C_1 \equiv C_2$ for this concept.

Example 3: Let $\mathcal{E}$ be an elliptic curve (over k), and let $F = P_1 \times \mathcal{E}$. Again, given a curve C on F, we can assign to C two degrees d and e, as the orders of the coverings

$$C \to P_1; \quad C \to \mathcal{E}$$

obtained by projecting. Both $d \geq 0$ and $e \geq 0$ and either $d > 0$ or $e > 0$.

Case i) $d = 0$: Then C is of the form $\Sigma_{i=1}^{e} P_i \times \mathcal{E}$, and all these C form a single e-dimensional linear system.

Case ii) $d > 0$: The set of all C of type (d, e) forms an irreducible $d(e + 1)$-dimensional family of curves, but it is not a linear system. Rather it is fibred by $d(e + 1)-1$-dimensional linear subfamilies.

Definition: Two curves C_1, C_2 are algebraically equivalent if C_1 and C_2 are both contained in one family of curves parametrized by a connected variety.

With this terminology, we can say that on $P_1 \times \mathcal{E}$, algebraic and linear equivalence differ. Another point to notice is that the dimension formula in Case ii) does not specialize to the dimensional formula in Case i) when $d = 0$: this is the phenomenon of superabundance.

Example 4: Let γ be a "generic" curve of genus 2, i.e., a double covering of $\mathbf{P}_1$ branched at six points with independent transcendental coordinates over the prime field (if char. $\neq 2$). Let F be the jacobian of γ. Recall that

(1) F is a non-singular algebraic surface,

(2) F is also an algebraic group,

(3) in a natural way, γ itself is a curve on F.

It turns out that every curve C on F is algebraically equivalent to a curve $d\gamma$, for a suitable positive integer d. Moreover, C is linearly equivalent to a suitable translation of $d\gamma$ (in the sense of the given group structure). The set of all curves algebraically equivalent to $d\gamma$ is an irreducible family of dimension $d^2 + 1$, and its linear sub-families have dimension $d^2 - 1$. In fact, one can define a map:

$$F \to \left[\frac{\text{all curves alg. equivalent to } d\gamma}{\text{linear equivalence}}\right]$$

where $a \mapsto$ image of $d\gamma$ under translation by a. In fact, this map factors as follows:

$$F \xrightarrow{\text{mult. by } d} F \xrightarrow{\text{bijection}} \left[\frac{\text{curves alg. equivalent to } d\gamma}{\text{linear equivalence}}\right]$$

This indicates a general point: the set [algebraic equivalence modulo linear equivalence], tends to be independent of the family of curves considered.

One should contrast this surface F with its "Kummer" counterpart K: this is defined as the double covering of $\mathbf{P}_2$ branched in a generic sextic curve (char. $\neq 2$). Here all curves are <u>linearly</u> equivalent to $d \cdot h$, where h is the inverse image of a line in $\mathbf{P}_2$, and the dimension of this family is $d^2 + 1$ (as above). It is similar to F also in that (a) $(\gamma^2) = 2$ on F, $(h^2) = 2$ on K [(D^2) denotes self-intersection—cf. Lecture 12], and (b) both F and K admit double differentials with neither zeros nor poles. This K is of the same type as the <u>quartic</u> surfaces in $\mathbf{P}_3$.

In fact, we have touched briefly on every class of algebraic surfaces admitting a double differential <u>with no zeros</u> (i.e., an anti-canonical linear system): for reasons stemming from Serre duality, the geometry on these surfaces is particularly simple. To bring out some further features of surfaces, we will discuss another rational surface:

Example 5: Let F be the surface obtained by blowing up two points P_1, P_2 in $\mathbf{P}_2$ [or by blowing up one point in $\mathbf{P}_1 \times \mathbf{P}_1$]. Let E_1 and E_2 be the rational curves which are the inverse images of P_1 and P_2 on F. Let ℓ be the line in $\mathbf{P}_2$ from P_1 to P_2, and let D be the curve on F which is the closure of the inverse image of $\ell - P_1 - P_2$. Then to every curve C on F, one can attach <u>three</u> characters k_1, k_2, and ℓ, where k_1, k_2 and ℓ are non-negative and not all zero; and the set of all curves with characters k_1, k_2, ℓ form the single linear system containing

$$k_1E_1 + k_2E_2 + \ell D$$

But unlike the situation on $P_1 \times P_1$, not all these systems are "good" systems of curves.

Case i) If $\ell \geq k_1$, $\ell \geq k_2$ and $k_1 + k_2 \geq \ell$, then none of the three curves E_1, E_2, or D is a component of all curves in the linear system containing $k_1E_1 + k_2E_2 + \ell D$, and this linear system has the predictable dimension:

$$(*) \qquad \frac{(\ell+1)(\ell+2)}{2} - \frac{(\ell-k_1)(\ell-k_1+1)}{2} - \frac{(\ell-k_2)(\ell-k_2+1)}{2} - 1$$

Case ii) If $\ell < k_1$, $\ell < k_2$, or $k_1 + k_2 < \ell$, then one of the three curves E_1, E_2, or D is a component of all the curves in question, and, in general, this family is also superabundant, i.e., its dimension is bigger than that predicted by (*).

Another way of telling the "good" from the "bad" systems of curves is this:

$$\left\{\begin{array}{l}\text{the system of curves}\\ \text{linearly equivalent}\\ \text{to } k_1E_1 + k_2E_2 + \ell D\\ \text{is the family of hyper-}\\ \text{plane sections of } F\\ \text{for some embedding of } F\\ \text{in } P_N\end{array}\right\} \Longleftrightarrow \begin{array}{l}\ell > k_1\\ \ell > k_2\\ k_1 + k_2 > \ell\end{array}$$

Here the condition on the left defines the notion: $k_1E_1 + k_2E_2 + \ell D$ is very ample.

With all this data before us, what questions emerge as the natural ones to pose in studying the curves on a general surface F ? I think four basic lines of study are suggested:

(i) the problem of Riemann-Roch: Given a curve C, to determine the dimension of the linear system of curves containing C. We shall see below that this is equivalent to the problem of computing

$$\dim H^0(\mathfrak{L})$$

where $\mathfrak{L}$ is a sheaf on F, locally isomorphic to the sheaf $\underline{o}_F$ of regular functions.

(ii) the problem of Picard: To describe the family of all algebraic deformations of a curve C modulo its linear subfamilies. It turns out that this quotient is independent of C, if C is good, and this quotient leads to the Picard scheme and/or variety.

(iii) Good vs. Bad curves: What makes C good and bad? One can ask when is C very ample, when is C superabundant, what are the really bad "exceptional" C which play the role of E_1, E_2 and D in Example 5 above? Particularly significant is the question of the "regularity of the adjoint" (= "Kodaira's vanishing theorem") cf. Lecture 14.

(iv) the components of the set of all curves C on F: Especially, what finiteness statements can be made? Examples are the theorem of the base of Neron and Severi, and the theorem that only a finite number of components represent curves of any given degree.

LECTURE 2

THE FUNDAMENTAL EXISTENCE PROBLEM AND TWO ANALYTIC PROOFS

We shall analyze problem ii) more closely. The real nature of the problem becomes clearer when one passes from curves to divisors. By a divisor on F we mean either a finite sum of irreducible, 1-dimensional subvarieties, with (positive or negative) multiplicity: $\Sigma\, n_i C_i$, $n_i \in \mathbb{Z}$, or a sheaf of fractional ideals, i.e., a coherent subsheaf of the constant sheaf $\underline{K}$:

$$\underline{K}(U) = \text{function field } k(F), \text{ all } U$$

(cf. Lecture 9 for precise definitions). The set of all divisors on F forms a group, which we denote $G(F)$. Put:

$G_a(F)$ = subgroup of divisors of the form $C_1 - C_2$, where C_1, C_2 are algebraically equivalent curves,

$G_\ell(F)$ = subgroup of divisors of the form $C_1 - C_2$, where $C_1 \equiv C_2$, or, equivalently, the subgroup of divisors of form (f), $f \in k(F)$.

Now if C is any curve on F, and $\{C_\alpha \mid \alpha \in S\}$ is the family of all curves algebraically equivalent to $C = C_0$, one can define a map:

$$S/\begin{matrix}\text{modulo linear}\\ \text{subfamilies}\end{matrix} \longrightarrow G_a(F)/G_\ell(F)$$

by mapping α to the divisor $C_\alpha - C_0$. One checks immediately that it is always injective, and it can be shown that for sufficiently "good" (?!) curves, it is surjective. For this reason, problem (ii) becomes independent of C, in most cases, and asks simply—what is the structure and dimension of the group $G_a(F)/G_\ell(F)$ invariantly attached to F ?

Again without proofs, we would like to mention the cohomological interpretation of these groups:

Let $\underline{o}^*$ = sheaf of units in the structure sheaf $\underline{o}$

$\underline{K}^*$ = sheaf of units in $\underline{K}$.

Then:

$$0 \to \underline{o}^* \to \underline{K}^* \to \underline{K}^*/\underline{o}^* \to 0$$

leads to: