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Sturm-Liouville Theory

Anton Zettl

American Mathematical Society

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Sturm-Liouville Theory

Preface

In 1836-1837 Sturm and Liouville published a series of papers on second order linear ordinary differential equations including boundary value problems. The influence of their work was such that this subject became known as Sturm-Liouville theory. The impact of these papers went well beyond their subject matter to general linear and nonlinear differential equations and to analysis generally, including functional analysis. Prior to this time the study of differential equations was largely limited to the search for solutions as analytic expressions. Sturm and Liouville were among the first to realize the limitations of this approach and to see the need for finding properties of solutions directly from the equation even when no analytic expressions for solutions are available.

Many thousands of papers, by Mathematicians, Physicists, Engineers and others, have been written since then. Yet, remarkably, this subject is an intensely active field of research today. Dozens of papers are published on Sturm-Liouville Problems (SLP) every year.

In 1910 Hermann Weyl published one of the most widely quoted papers in analysis [**607**]. Just as the 1836-37 papers of Sturm and Liouville started the study of *regular* SLP, the 1910 paper of Weyl initiated the investigation of *singular* SLP. The development of quantum mechanics in the 1920's and 1930's, the proof of the general spectral theorem for unbounded self-adjoint operators in Hilbert space by von Neumann and Stone, and the fundamental work of Titchmarsh [**573**] provided some of the motivation for further investigations into the spectral theory of Sturm-Liouville operators.

The purpose of this monograph is twofold: (i) to give a modern survey of some of the basic properties of the Sturm-Liouville equation and (ii) to bring the reader to the forefront of knowledge on some aspects of SLP.

On numerous occasions I have been asked: Where can I find a readable introduction to Sturm-Liouville problems? Although the subject matter of SLP is briefly discussed in many books, these discussions tend to be sketchy, particularly in the singular case. Even for the regular case, a general discussion of separated and coupled self-adjoint boundary conditions is not easy to find in the existing literature. We hope that this monograph will serve as a readable introduction to SLP and, at the same time, provide an up to date account of some parts of this fascinating subject.

A major stimulus for the writing of this monograph was the authors' collaboration with Paul Bailey and Norrie Everitt on the development of the fortran code SLEIGN2 for the numerical computation of eigenvalues and eigenfunctions of *regular and singular, separated and coupled, self-adjoint* SLP. All nine files of the SLEIGN2 software package as well as numerous related papers can be downloaded from: http:/www.math.niu.edu/~zettl/SL2/.

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The code is designed to be used by novice and expert alike. It comes with a user friendly interface and an interactive help tutorial. When used with some theoretical results in papers, some of which are available from the web page just mentioned, SLEIGN2 can also be used to approximate parts of the essential (continuous) spectrum, e.g. the starting point of the essential spectrum, the first few spectral bands or gaps, etc.

Although the subject of Sturm-Liouville problems is over 170 years old, a surprising number of the results surveyed here are of recent origin, some were published within the last couple of years and a few are not in print at the time of this writing.

The book is divided into five parts. Part I deals with existence and uniqueness questions for initial value problems including the continuous and differentiable dependence of solutions on all the parameters of the problem. Regular two-point boundary value problems are discussed in Part II, non-self-adjoint problems in Chapter 3, classical (right-definite) self-adjoint problems in Chapter 4 and left-definite and indefinite problems in Chapter 5. Oscillation, the limit-point/limit-circle dichotomy and singular initial value problems are covered in Part III. Singular self-adjoint, non-self-adjoint, right-definite, left-definite and indefinite problems are studied in Part IV. Part V contains chapters on notation, topics not covered, the two-interval theory of boundary value problems and a chapter on examples. These examples have been chosen to illustrate the depth and diversity of Sturm-Liouville theory.

When I started this project it was my intention to provide detailed proofs of all results and to give an elementary proof whenever possible. But I soon realized that this task was beyond my energy level. So I have compromised by providing some detailed proofs, as elementary as possible, and readable references to all proofs not given. Many of these references can be found on the web address mentioned above.

I have been privileged to work with many mathematicians and am grateful to all of them. Special thanks go to my co-authors: F.V. Atkinson, P.B. Bailey, J. Billingham, B.M. Brown, X. Cao, E.A. Coddington, R. J. Cooper, H.I. Dwyer, M.S.P. Eastham, W. Eberhard, W.D. Evans, W.N. Everitt, Z.M. Franco, G. Freiling, H. Frentzen, M. Giertz, J. Goldstein, J. Gunson, K. Haertzen, D.B. Hinton, H.G. Kaper, R.M. Kauffman, A.C. King, L. Kong, Q. Kong, M.K. Kwong, A.M. Krall, G. Leaf, H. Lekkerkerker, Q. Lin, L. Littlejohn, M. Marletta, D.K.R. McCormack, M. Möller, H.-D. Niessen, D. Race, B.S. Garbow, T.T. Read, J. Ridenhour, C. Shubin, G. Stolz, H. Volkmer, J. Weidmann, J.S.W. Wong, and H. Wu. In particular I thank Paul Bailey for introducing me to the wonderful world of computing, for his seemingly infinite patience in writing, debugging and improving the SLEIGN2 code.

I am greatly indebted to my colleagues and friends Qingkai Kong, Man Kam Kwong, and Hongyou Wu for many hundreds of hours of enjoyable and productive collaboration.

I am especially grateful to my friend and collaborator for more than a quarter century, W.N. Everitt. Norrie's characteristically careful and exacting criticisms have resulted in numerous improvements not only of the contents but also the presentation of this monograph. Moreover, I like to think that some of his infinite enthusiasm for Mathematics and his masterful writing style have rubbed off on me.

Special thanks go to the two anonymous referees for their thorough reading of the manuscript. Their corrections, suggestions and criticisms have resulted in numerous improvements.

Last, but certainly not least, I thank my wife Sandra for her help with the database for the references and, especially, for helping with the hardware and software problems that arose during the typing of this manuscript with Scientific Workplace (SWP). Also for her tolerance and understanding during this and many other Mathematics projects.

The world of Mathematics is full of wonders and of mysteries, at least as much so as the physical world.

Contents

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Part 1

Existence and Uniqueness Problems

da Vinci, Leonardo (1452-1519)

He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast.

No human investigation can be called real science if it cannot be demonstrated mathematically.

Stewart, Ian

The successes of the differential equation paradigm were impressive and extensive. Many problems, including basic and important ones, led to equations that could be solved. A process of self-selection set in, whereby equations that could not be solved were automatically of less interest than those that could.

Does God Play Dice? The Mathematics of Chaos, Blackwell, Cambridge, MA, 1989.

CHAPTER 1

First Order Systems

H. S. Wall

> The Mathematician is an artist whose medium is the mind and whose creations are ideas.

1. Introduction

This chapter is devoted to the study of basic properties of first order systems of general dimension n. Although our primary interest is in the case $n = 2$ we include the higher order case since it can be studied with basically the same methods.

Notation. An open interval is denoted by (a, b) with $-\infty \le a < b \le \infty$; $[a, b]$ denotes the closed interval which includes the left endpoint a and the right endpoint b, regardless of whether these are finite or infinite, $\mathbb{R}$ denotes the reals, $\mathbb{C}$ the complex numbers, and

$$\mathbb{N} = \{1, 2, 3, \ldots\},\ \mathbb{N}_0 = \{0, 1, 2, \ldots\},\ \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

For any interval J of the real line, open, closed, half open, bounded or unbounded, by $L(J, \mathbb{C})$ we denote the linear manifold of complex valued Lebesgue measurable functions y defined on J for which

$$\int_a^b |y(t)|\, dt \equiv \int_J |y(t)|\, dt \equiv \int_J |y| < \infty.$$

The notation $L_{loc}(J, \mathbb{C})$ is used to denote the linear manifold of functions y satisfying $y \in L([\alpha, \beta], \mathbb{C})$ for all compact intervals $[\alpha, \beta] \subseteq J$. If $J = [a, b]$ and both of a and b are finite, then $L_{loc}(J, \mathbb{C}) = L(J, \mathbb{C})$. Also, we denote by $AC_{loc}(J)$ the collection of complex-valued functions y which are absolutely continuous on all compact intervals $[\alpha, \beta] \subseteq J$. The symbols $L(J, \mathbb{R})$ and $L_{loc}(J, \mathbb{R})$ are defined similarly.

For a given set S, $M_{n,m}(S)$ denotes the set of $n \times m$ matrices with entries from S. If $n = m$ we write $M_n(S) = M_{n,n}(S)$; also if $m = 1$ we sometimes write S^n for $M_{n,1}(S)$. The norm of a constant matrix as well as the norm of a matrix function P is denoted by $|P|$. This may be taken as

$$|P| = \sum |p_{ij}|.$$

2. Existence and Uniqueness of Solutions

Definition 1.2.1 (Solution). *Let J be any interval, open, closed, half open, bounded or unbounded; let $n, m \in \mathbb{N}$, let $P : J \to M_n(\mathbb{C})$, $F : J \to M_{n,m}(\mathbb{C})$. By a solution of the equation*

$$Y' = PY + F \text{ on } J$$

we mean a function Y from J into $M_{n,m}(\mathbb{C})$ which is absolutely continuous on all compact subintervals of J and satisfies the equation a.e. on J. A matrix function is absolutely continuous if each of its components is absolutely continuous.

THEOREM 1.2.1 (Existence and uniqueness). *Let J be any interval, open, closed, half open, bounded or unbounded; let $n, m \in \mathbb{N}$. If*

$$P \in M_n(L_{loc}(J, \mathbb{C})) \tag{1.2.1}$$

and

$$F \in M_{n,m}(L_{loc}(J, \mathbb{C})) \tag{1.2.2}$$

then every initial value problem (IVP)

$$Y' = PY + F \tag{1.2.3}$$

$$Y(u) = C, \ u \in J, \ C \in M_{n,m}(\mathbb{C}) \tag{1.2.4}$$

has a unique solution defined on all of J. Furthermore, if C, P, F, are all real-valued, then there is a unique real valued solution.

PROOF. We give two proofs of this important theorem; the second one is the standard successive approximations proof. As we will see later the analytic dependence of solutions on the spectral parameter λ follows more readily from the second proof than the first.

For both proofs we note that if Y is a solution of the IVP (1.2.3), (1.2.4) then an integration yields

$$Y(t) = C + \int_u^t (PY + F), \ t \in J. \tag{1.2.5}$$

Conversely, every solution of the integral equation (1.2.5) is also a solution of the IVP (1.2.3), (1.2.4).

Choose c in J, $c \neq u$. We show that (1.2.5) has a unique solution on $[u, c]$ if $c > u$ and on $[c, u]$ if $c < u$. Assume $c > u$. Let

$$B = \{Y : [u, c] \to M_{n,m}(\mathbb{C}), \ Y \ continuous\}.$$

Following Bielecki [**67**] we define the norm of any function $Y \in B$ to be

$$\|Y\| = \sup\left\{\exp\left(-K \int_u^t |P(s)|\, ds\right) |Y(t)|, \ t \in [u, c]\right\}, \tag{1.2.6}$$

where K is a fixed positive constant $K > 1$. It is easy to see that with this norm B is a Banach space. Let the operator $T : B \to B$ be defined by

$$(TY)(t) = C + \int_u^t (PY + F)(s)\, ds, \ t \in [u, c], \ Y \in B. \tag{1.2.7}$$

Then for $Y, Z \in B$ we have

$$|(TY)(t) - (TZ)(t)| \leq \int_u^t |P(s)||Y(s) - Z(s)| ds$$

and hence

$$\exp\left(-K\int_u^t |P(s)|\,ds\right)|(TY)(t)-(TZ)(t)|$$
$$\leq \|Y-Z\|\int_u^t |P(s)|\exp\left(-K\int_s^t |P(r)|\,dr\right)ds$$
$$\leq \frac{1}{K}\,\|Y-Z\|.$$

Therefore

$$\|TY-TZ\|\leq \frac{1}{K}\,\|Y-Z\|.$$

From the contraction mapping principle in Banach space it follows that the map T has a unique fixed point and therefore the IVP (1.2.3), (1.2.4) has a unique solution on $[u,c]$. The proof for the case $c<u$ is similar; in this case the norm of B is modified to

$$\|Y\|=\sup\{\exp\left(K\int_u^t |P(s)|\,ds\right)|Y(t)|,\ t\in[c,u]\}.$$

Since there is a unique solution on every compact subinterval $[u,c]$ and $[c,u]$ for $c\in J$, $c\neq u$ it follows that there is a unique solution on J. To establish the furthermore part take the Banach space of real-valued functions and proceed similarly. This completes the first proof.

For the second proof we construct a solution of (1.2.5) by successive approximations. Define

$$Y_0(t)=C,\ Y_{n+1}(t)=C+\int_u^t (PY_n+F),\ t\in J,\ n=0,1,2,\ldots \tag{1.2.8}$$

Then Y_n is a continuous function on J for each $n\in N_0$. We show that the sequence $\{Y_n : n\in N_0\}$ converges to a function Y uniformly on each compact subinterval of J and that the limit function Y is the unique solution of the integral equation (1.2.5) and hence also of the IVP (1.2.3), (1.2.4). Choose $b\in J$, $b>u$ and define

$$p(t)=\int_u^t |P(s)|\,ds,\ t\in J;\ B_n(t)=\max_{u\leq s\leq t}|Y_{n+1}(s)-Y_n(s)|,\ u\leq t\leq b. \tag{1.2.9}$$

Then

$$Y_{n+1}(t)-Y_n(t)=\int_u^t P(s)\,[Y_n(s)-Y_{n-1}(s)]\,ds,\ t\in J,\ n\in\mathbb{N}. \tag{1.2.10}$$

From this we get

$$|Y_2(t)-Y_1(t)|\leq B_0(t)\int_u^t |P(s)|\,ds=B_0(t)\,p(t)\leq B_0(b)\,p(b),\ u\leq t\leq b. \tag{1.2.11}$$

$$|Y_3(t)-Y_2(t)|\leq \int_u^t |P(s)|\,|Y_2(s)-Y_1(s)|\,ds\leq \int_u^t |P(s)|\,B_0(s)\,p(s)\,ds$$
$$\leq B_0(t)\int_u^t |P(s)|\,p(s)\,ds\leq B_0(b)\,\frac{p^2(t)}{2!}$$
$$\leq B_0(b)\,\frac{p^2(b)}{2!},\ u\leq t\leq b.$$

From this and mathematical induction we get

$$|Y_{n+1}(t) - Y_n(t)| \le B_0(b)\,\frac{p^n(b)}{n!}\ ,\ u \le t \le b.$$

Hence for any $k \in N$

$$\begin{aligned}|Y_{n+k+1}(t) - Y_n(t)| &\le |Y_{n+k+1}(t) - Y_{n+k}(t)| + |Y_{n+k}(t) - Y_{n+k-1}(t)| + \\ &\quad \ldots + |Y_{n+1}(t) - Y_n(t)| \\ &\le B_0(b)\,\frac{p^n(b)}{n!}[1 + \frac{p(b)}{n+1} + \frac{p^2(b)}{(n+2)(n+1)} + \ldots].\end{aligned}$$

Choose m large enough so that $p(b)/(n+1) \le 1/2$, then $p^2(b)/((n+2)(n+1)) \le 1/4$, etc. when $n > m$ and the term in brackets is bounded above by 2. It follows that the sequence $\{Y_n : n \in N_0\}$ converges uniformly, say to Y, on $[u, b]$. From this it follows that Y satisfies the integral equation (1.2.5) and hence also the IVP (1.2.3), (1.2.4) on $[u, b]$.

To show that Y is the unique solution assume Z is another one; then Z is continuous and therefore $|Y - Z|$ is bounded, say by $M > 0$ on $[u, b]$. Then

$$|Y(t) - Z(t)| = \left|\int_u^t P(s)[Y(s) - Z(s)]\,ds\right| \le M\int_u^t |P(s)|\,ds \le M\,p(t),\ u \le t \le b.$$

Now proceeding as above we get

$$|Y(t) - Z(t)| \le M\,\frac{p^n(t)}{n!} \le M\,\frac{p^n(b)}{n!}\ ,\ u \le t \le b,\ n \in \mathbb{N}.$$

Therefore $Y = Z$ on $[u, b]$. There is a similar proof for the case when $b < u$. This completes the second proof. □

It is interesting to observe that the initial approximation $Y_0(t) = C$ can be replaced with $Y_0(t) = G(t)$ for any continuous function G without any essential change in the proof.

To study the dependence of the unique solution on the parameters of the problem we introduce a convenient notation. Let J be an interval. For each $P \in M_n(L_{loc}(J, \mathbb{C}))$, each $F \in M_{n,m}(L_{loc}(J, \mathbb{C}))$, each $u \in J$ and each $C \in M_{n,m}(\mathbb{C})$ there is, according to Theorem 1.2.1, a unique $Y \in M_{n,m}(AC_{loc}(J))$ such that $Y' = PY + F$, $Y(u) = C$. We use the notation

$$Y = Y(\cdot, u, C, P, F) \tag{1.2.12}$$

to indicate the dependence of the unique solution Y on these quantities. Below, if the variation of Y with respect to some of the variables u, C, P, F is studied while the others remain fixed we abbreviate the notation (1.2.12) by dropping those quantities which remain fixed. Thus we may use $Y(t)$ for the value of the solution at $t \in J$ when u, C, P, F are fixed or $Y(\cdot, u)$ to study the variation of the solution function Y with respect to $u, Y(\cdot, P)$ to study Y as a function of P, etc.

THEOREM 1.2.2 (Rank invariance). *Let $J = (a, b)$, and assume that $P \in M_n(L_{loc}(J, \mathbb{C}))$. If Y is an $n \times m$ matrix solution of*

$$Y' = PY \text{ on } J, \tag{1.2.13}$$

then we have

$$\operatorname{rank} Y(t) = \operatorname{rank} Y(u), \quad t, u \in J. \tag{1.2.14}$$

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