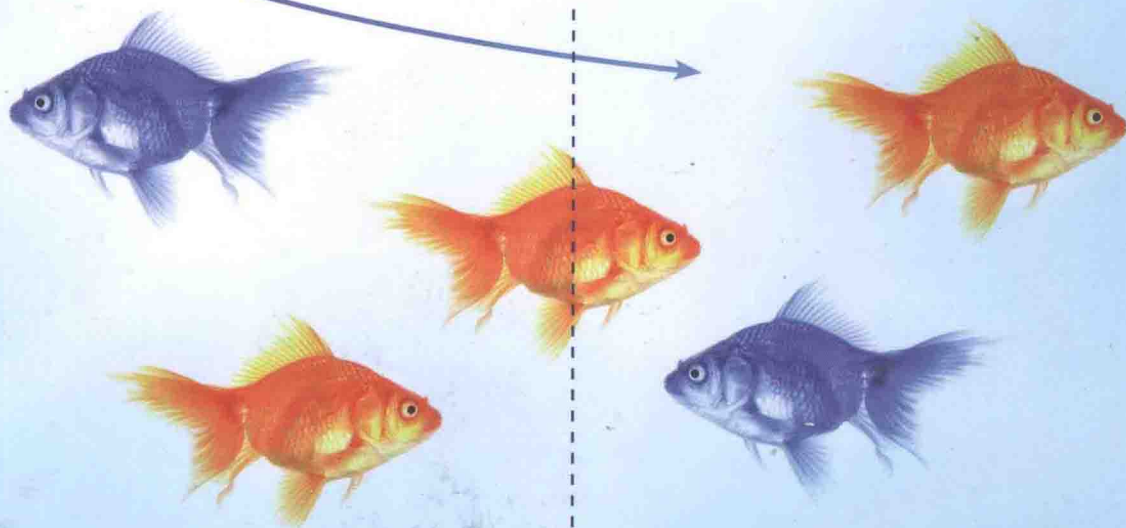


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# Relativity Made Relatively Easy



Andrew M. Steane

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ANDREW M. STEANE



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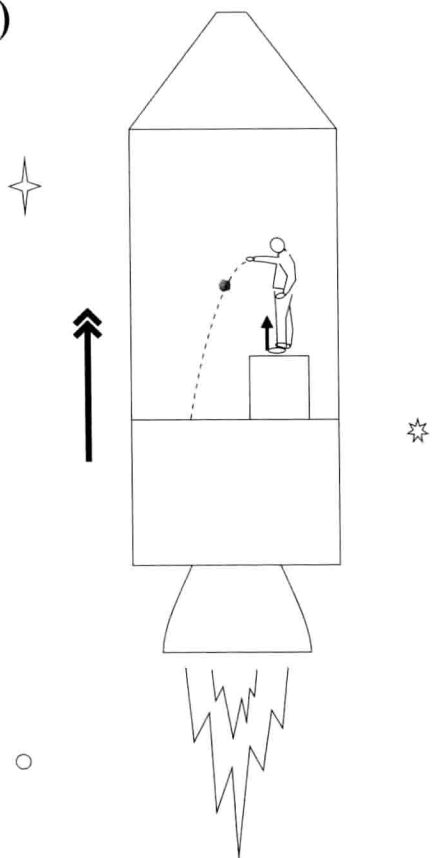
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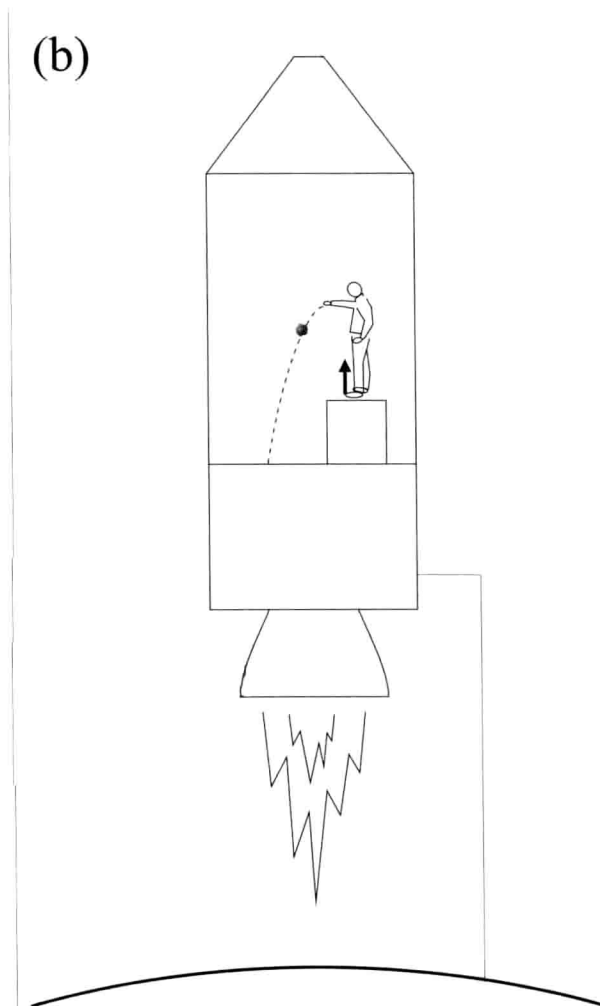
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(b)



# Dedication

This book is dedicated to Derek Stacey and David Lucas: two physicists from whose generosity, encouragement and example I have greatly benefited.

# Preface

The aim of this book is to help people understand and enjoy the world in which we all live. It is written for the undergraduate student of physics, and is intended to teach. The text presents an extensive study of Special Relativity, and a (gentle, but exact) introduction to General Relativity. It is not intended to be the first introduction to Special Relativity for most students, although for a bright student it could function as that. Therefore basic ideas such as time dilation and space contraction are recalled but not discussed at length. However, I think it is also beneficial to have a thorough discussion of those concepts at as simple a level as possible, so I have provided one in another book called *The Wonderful World of Relativity*. The present book is self-contained and does not require knowledge of the first one, but a more basic text such as *The Wonderful World* or something similar is recommended as a preparation for this book.

The book has two more specific aims. The first is to allow an undergraduate physics course to extend somewhat further and wider in this area than has traditionally been the case, while ensuring that the mainstream of students can still handle the material; the second is to show how physics ‘works’ more generally and to act as a prelude to advanced topics such as classical and quantum field theory. The title *Relativity Made Relatively Easy* is therefore playful, yet serious. The text aims to make manageable what would otherwise be regarded as hard; to make derivations as simple as possible and physical ideas as transparent as possible. It is intended to teach, and therefore little prior mathematical knowledge is assumed. Although spacetime and relativity are the main themes, physical ideas such as fields and flow, symmetry and stress are expounded along the way. These ideas connect to other areas such as hydrodynamics, electromagnetism, and particle physics. The present volume covers Special Relativity thoroughly except for spinors and Lagrangian density methods for fields, and it introduces General Relativity with the minimum of mathematical apparatus required to acquire correct ideas and quantitative results for static metrics. The affine connection (Christoffel symbol), for example, is not needed in order to achieve this. A second volume will extend the treatment of General Relativity somewhat more thoroughly, and will also introduce cosmology, spinors and some field theory (which explains the occasional mention of ‘volume 2’ in the present text).

Although many universities are now extending their core coverage of Relativity, and the present volume is intended to meet that need, a few sections go further than most undergraduate courses will want to go. These are intended to fill the gap between undergraduate and graduate study, and to offer general reading for the professional physicist.

The exercises are an integral part of the text, and are of three types. Some are examples to build familiarity, some introduce formulae or results that fill out or complete the main text, and some build physical intuition and a sound grasp of the big ideas.

Sections or chapters with a \* in the heading can be omitted or skimmed at first reading.



# Acknowledgements

I have, of course, learned Relativity mostly from other people. All writers in this area have learned from the pioneers of the subject, especially Einstein, Lorentz, Maxwell, Minkowski, and Poincaré. I am also indebted to tutors such as N. Stone and W. S. C. Williams at Oxford University, and to authors who have preceded me—especially texts by (in alphabetical order) J. Binney, T.-P. Cheng, A. Einstein, R. Feynman, A. P. French, R. d’Inverno, J. D. Jackson, H. Muirhead, W. Rindler, F. Rohrlich, R. Shankar, E. F. Taylor, J. A. Wheeler, and W. S. C. Williams. I thank A. Barr for some comments and suggestions.

Several sections of the book compress, extend, clarify or combine treatments from other authors. I have not traced the ancestry of all the material, but have indicated cases where I quote an approach or argument that I think might not be regarded as common currency. I have also developed many ideas independently, but naturally after a century of discussion, a significant proportion of such ideas must be rediscoveries.

Einstein emphasized the need to think of a reference frame in physical and not abstract terms, as a physical entity made of rods and clocks. As a student I resisted this idea, feeling that a more abstract idea, liberated from mere matter, must be superior. I was wrong. The whole point of Relativity is to see that abstract notions of space and time are superfluous and misleading.

Feynman offered very useful guidance on how to approach things simply while retaining rigour. Readers familiar with his, Leighton, and Sands’ ‘Lectures on Physics’ will recognize its influence on Chapter 10. I have filled in mathematical methods in order to allow the student to tackle example calculations, and to make Chapter 11 possible.

I learned a significant number of detailed points from Rindler’s work, and my contribution has been to clarify where possible. Appendix D and section 16.2.1 re-present arguments I found in his book, with more comfort and explanation for the reader, and section 13.1 follows his line of argument, which I could not improve.

Most of the exercises are either original or significantly reconfigured, or are standard short problems. However, some have been largely copied from elsewhere—a § indicates those from W. Rindler, and §§ those from W. S. C. Williams. I thank these authors for their permission. Wolfgang Rindler has produced many excellent exercises in several books; I thank him for allowing a selection of them to adorn this book.

I thank my family, who accepted the compromises on home life which I made in order to work on the book.

Finally, my special thanks go to the Physics Department of Oxford University, its ‘Ion Traps and Quantum Computing’ research group, and David Lucas and Charles de Bourcy. The University allowed an atmosphere of academic freedom and trust that made it possible to devote time to this book, the research group, under David’s guidance, accepted my long absences in a generous spirit, and Charles kindly read and checked part I of the manuscript with great care and insight. Any remaining defects are my own responsibility.

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# **Part I**

## **The relativistic world**



