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Mathematics of Finance

Proceedings of an AMS–IMS–SIAM
Joint Summer Research Conference
on Mathematics of Finance
June 22–26, 2003
Snowbird, Utah

George Yin
Qing Zhang
Editors

American Mathematical Society

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Preface

This volume contains 31 papers, based on the invited talks given at the 2003 AMS-IMS-SIAM Joint Summer Research Conference in the Mathematical Sciences: Mathematics of Finance, held in Snowbird, Utah, June 22–26, 2003. This was the first ever conference on mathematics of finance jointly sponsored by AMS, IMS, and SIAM.

Financial mathematics is a rapidly expanding field. It involves a wide spectrum of techniques that go far beyond the traditional applied mathematics. Research in mathematics of finance has witnessed tremendous progress in recent years. The Black-Scholes model and its various extensions for pricing of options have had an influential impact on financial practice and led to a revolution in the financial industry. The introduction of stochastic analysis and stochastic control techniques has resulted in a number of important advances. To name just a few, they include the studies of valuation of contingent claims in complete and incomplete markets, consumption-investment models with or without constraints, portfolio management for institutional investors such as pension funds and banks, and risk assessment and management using financial derivatives. These applications, on the other hand, require and stimulate many new and exciting theoretical discoveries. As a major impetus to the development of financial management and economics, research in mathematics of finance has had a major impact on the global economy. Moreover, the development of mathematics of finance has created a large demand for mathematics graduates at both Master and Ph.D. levels in the financial industry, resulting in the introduction of this topic in the curriculum of mathematical sciences departments of many universities. The rapid progress has necessitated communication and networking among researchers in different disciplines. This summer research conference provided us with an excellent and timely opportunity. It brought together researchers from mathematical sciences, finance, economics, and engineering, and financial industry to review and to update the recent advances, and to identify future directions of mathematics of finance.

The scientific program of the conference consisted of 42 invited talks, a poster session, and a panel discussion on research and education. While recent progress has been surveyed, reviewed, and substantially updated, new ideas, models, methods, and techniques have been explored. The invited speakers presented a broad spectrum of problems, models, and results involving modeling, estimation, optimization, control, risk assessment and management, contingent claim pricing, dynamic hedging, and financial derivative design. Valuation of contingent claims remains the centerpiece of modern financial theory. Its key components include financial market modeling and dynamic hedging. While the Black-Scholes models have been widely used in characterizing movements of financial markets for decades, it has been recognized that their utility is limited because they assume deterministic rates of return and volatility, and because they ignore many aspects of the markets. In the past few years, various attempts (including stochastic volatility, jump diffusions, and hybrid market models) have emerged to modify and generalize the Black-Scholes models. Optimal portfolio management uses a stochastic control approach. Originating from Merton's pioneering work, it continues to have an important role in finance theory. The objective is to allocate financial assets dynamically among risky and fixed-income investments with the goal of maximizing expected overall return of consumption measured by some utility function. A closed-form solution is possible only for the simplest models. Typically, optimal investment and consumption control policies must be found by solving a partial differential equation of Hamilton-Jacobi-Bellman type. However, nonlinearities make numerical implementations difficult, and efficient schemes are needed. Other difficult issues include, for example, mathematical model selections and choices of utility functions. Financial risk management has attracted growing attention in recent years. Such devastating events as the Long-Term Capital Management default and the Enron bankruptcy shook the financial world. It has become clear that there is an urgent need for further research on corporation credit risks as well as the possibility to hedging these risks using financial derivatives. As a result, one of the emerging research topics is the study of credit risk management.

As an archive, this volume presents some of the highlights of the conference. It collects papers covering a broad spectrum of topics in mathematical finance; all papers have been refereed. The organizing committee consisted of Wendell H. Fleming (Brown University), Jean-Pierre Fouque (North Carolina State University), George Papanicolaou (Stanford University), Bozenna Pasik-Duncan (University of Kansas), Stanley R. Pliska (University of Illinois at Chicago), Ronnie Sircar (Princeton University), George Yin (Wayne State University, Chair), and Qing Zhang (University of Georgia, Co-chair). It was supported in part by the National Science Foundation.

Without the encouragement, help, and assistance of many individuals, the conference could not have taken place. We thank the invited speakers, the panelists, the poster presenters, and all invitees for making the conference a successful event; we thank the members of the organizing committee for their help, advice, and suggestions. Our thanks go to the AMS-IMS-SIAM Committee on Summer Research Conferences in the Mathematical Sciences, in particular, Thomas DiCiccio, chair of the committee, and James Maxwell of the AMS, who helped us shape the conference and provided us with valuable comments and suggestions in the preparation of the conference. We are especially grateful to Donna Salter and Wayne Drady for their constant and tireless help during the preparation of the conference as on-site management. The assistance from Sergei Gelfand, Christine Thivierge, and the AMS professionals during the preparation of this volume is also gratefully acknowledged. Finally, we are thankful to the National Science Foundation for supporting the Summer Research Conference.

George Yin and Qing Zhang

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Credit Barrier Models in a Discrete Framework

Claudio Albanese and Oliver X. Chen

ABSTRACT. We formulate credit barrier models in the framework of jump processes with absorption on a discrete lattice. The lattice model is formulated in terms of finite state Markov processes related to the Hahn family of hypergeometric polynomials. The continuous limit we obtain as the lattice spacing goes to zero corresponds to the Jacobi process. The model is designed to relate real-world and risk neutral measures.

1. Introduction

Credit barrier models are derivative pricing models for credit sensitive instruments. The underlying is a credit quality variable with the meaning of distance to default, a measure of an obligor's leverage relative to the volatility of its asset values. The first models in this class appeared in working papers and internal documents [**HLPQ99**], [**GH01**], [**DJ02**] and made their way to the open literature in articles by Hull and White [**HW01**] and Avellaneda and Zhu [**AZ01**]. This first generation of credit barrier models involves estimations against a single spread curve. Real world estimates instead are applied to barrier models of the Merton type used for risk management applications as in the CreditMetrics^{TM} technical document by Gupton et al. [**GFB97**]. A new class of credit barrier models was introduced by the authors in [**ACCZ03**] and [**AC03**] in an attempt to reconcile the real-world and the risk-neutral measure. In this new class, the estimation framework is extended to include a more comprehensive set of statistical data such as historical migration rates, default frequencies over several time horizons and aggregate spread curves across all ratings. Within the extended framework, one obtains metrics for relative liquidity spreads across credit ratings. One also obtains a new methodology to extrapolate implied migration rates; in [**AC03**] we compare with

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earlier methods proposed by Jarrow, Lando and Turnbull [**JLT97**], and by Kijima and Komoribayashi [**KK98**].

The present paper recasts the model in [**AC03**] in a new mathematical framework which has much greater numerical efficiency. While the previous model was constructed upon CIR processes, here we start from a deformation of this process called the Jacobi process and a discretization thereof that we refer to as the Hahn process. This discretization is accomplished by approximating the continuous process by a birth-death process on a finite lattice, in continuous time. The Jacobi and Hahn processes are both integrable in terms of orthogonal hypergeometric polynomials (see, for example, [**Sch00**], [**NSUag**] and [**KS98**]). Hahn processes in particular are used to build a new class of lattice models in a recent work by Albanese and Kuznetsov [**AK03**]. These lattice models are characterized by the property that node-to-node transition probabilities are computed analytically in such a way that prices of European style options do not depend on how the time nodes are chosen. In this article, to extend this framework to the case of credit barrier models, we allow for absorption to be present at the boundary without compromising analytic solvability in closed form. Since these numerical schemes are flexible enough to accommodate Bochner subordinators, the models also extend to the case where the continuous limit process is a subordinated diffusion with jumps.

The relevant literature on applications of the theory of orthogonal polynomials to stochastic processes includes earlier works by Ledermann and Reuter [**LR54**] and Karlin and McGregor [**KM57**], who studied the spectral theory of birth-death processes. Karlin and McGregor [**KM61**] considered the dual Hahn polynomials in modeling a birth-death process with application to a problem in genetics. A more recent analysis of orthogonal polynomials in the context of birth-death processes is given by van Doorn [**vD03**]. Cases with absorption and explicit time dependency are considered by Lenin, et al [**LPSvD00**]. Lorente [**Lor03**] and [**Alb03**] examine integrable systems represented by orthogonal polynomials of discrete variable in classical and quantum physics. Finally, jump processes constructed in terms of orthogonal polynomials are studied by Schoutens [**Sch00**].

The rest of the paper proceeds as follows: section 2 will outline the procedure in the case of a continuous diffusion with the Jacobi process as the underlying process as opposed to the CIR process used in [**ACCZ03**] and [**AC03**]. Section 3 will give a full description of the discretization scheme and lays out the procedure for constructing a model. The risk-neutralizing procedure is not exactly analogous to the previous papers, but differs in order to take full advantage of the new discretization scheme. Section 4 concludes the paper.

2. Continuous case

We present the procedure for the Jacobi process in order to draw parallels to the previous CIR process studied in [**ACCZ03**] and [**AC03**]. We elected to change the underlying process from the CIR process to the Jacobi process because the Jacobi process is restricted to a finite interval. As will be seen in the following section, this leads to a finite number of basis eigenvectors upon discretization.

In order to model the real-world credit process, we require a local martingale defined on an interval I which is required to be bounded below. Without loss of generality, we set the lower boundary to be zero. We require zero to be an absorbing boundary, and consider the credit process y_t to have defaulted if absorption at zero occurs. Further, suppose that there are K different credit rating classes that an obligor can be in. We partition I into intervals $I_1, \ldots, I_K$ sequenced such that if $y_1 \in I_j$ and $y_2 \in I_k$ with $j < k$ then $y_1 < y_2$. Then, if at time t the credit process y_t of a given reference name is in the interval I_k, we consider that reference name to be in the k^{th} credit rating class.

2.1. Underlying process. In order to obtain an integrable local martingale, we first require an underlying stochastic process which in this case we choose to be the Jacobi process. In general, a stochastic process is described as being a Jacobi process if it satisfies the stochastic differential equation:

$$dx_t = (a - bx_t)dt + \nu_0\sqrt{(x_t - c)(d - x_t)}dW_t.$$

Since we utilize the Jacobi process only as an underlying process for a local martingale process, we are free to specialize to the process:

$$dx_t = (\alpha + 1 - (\alpha + \beta + 2)x_t)dt + \sqrt{2(1 - x_t)x_t}dW_t.$$

Here, $\alpha, \beta > -1$ and the process is defined on the interval $(0, 1)$. Any further freedom that is required for calibrating the process can be obtained by adjusting the subsequent measure change and transformation to obtain the local martingale process.

The infinitesimal generator of this process is given by the differential operator:

$$\mathcal{L} = (1 - x)x\frac{d^2}{dx^2} + [\alpha + 1 - (\alpha + \beta + 2)x]\frac{d}{dx}. \tag{2.1}$$

The probability kernel $u_t(x_0, x)$ for this process is the solution to the backward Kolmogorov equation:

$$\mathcal{L}u_t(x_0, x) = -\frac{d}{dt}u_t(x_0, x) \tag{2.2}$$

where $\mathcal{L}$ acts on the first variable x_0, with initial condition:

$$u_0(x_0, x) = \delta(x - x_0).$$

We note here that there is a representation of the probability kernel as an infinite sum of Jacobi polynomials, which are a family of orthogonal polynomials. However, as the continuous case is not the focus of the present paper, we will not give this expression. The analogous representation is given in the discrete case.

2.2. Constructing a local martingale process. In order to construct the measure change and transformation, we require two linearly independent solutions f_1 and f_2 of the eigenvalue equation:

$$\mathcal{L}f = \rho f \tag{2.3}$$

for some real number ρ. Suppose we can find two coefficients c_1 and c_2 such that the linear combination

$$g = c_1 f_1 + c_2 f_2$$

is strictly positive. Then, it is shown in [**AK03**] that for any choice of the coefficients c_3 and c_4 such that $c_1c_4 - c_2c_3 \neq 0$, the transformation:

$$Y = \frac{c_3 f_1 + c_4 f_2}{g} - \lim_{x\to 0+} \frac{c_3 f_1(x) + c_4 f_2(x)}{g(x)} \tag{2.4}$$

is invertible with inverse $Y^{-1} = X$ and the function

$$\tilde{u}_t(y_0, y) = e^{-\rho t} X'(y) \frac{g\left(X(y)\right)}{g(X(y_0))} u_t(X(y_0), X(y)) \tag{2.5}$$

is the probability kernel for a local martingale stochastic process with infinitesimal generator:

$$\tilde{\mathcal{L}} = X' g^{-1}(X)(\mathcal{L} - \rho) g(X)$$

and volatility function:

$$\sigma(y) = \sqrt{2X(y)(1 - X(y))} Y'(X(y)). \tag{2.6}$$

Notice that equation (2.3) is a hypergeometric differential equation, with general solutions in terms of the Gauss hypergeometric function:

$$f(x) = A\ {}_2F_1(a, b; \alpha + 1|x) + Bx^{-\alpha}\ {}_2F_1(a - \alpha, b - \alpha; 1 - \alpha|x) \tag{2.7}$$

where the parameters a and b are given by:

$$\begin{aligned} a &= \frac{(\alpha + \beta + 1) + \sqrt{(\alpha + \beta + 1)^2 - 4\rho}}{2} \\ b &= \frac{(\alpha + \beta + 1) - \sqrt{(\alpha + \beta + 1)^2 - 4\rho}}{2}. \end{aligned}$$

The general hypergeometric function ${}_pF_q$ is given in equation (A.2). Thus, we are free to choose

$$\begin{aligned} f_1(x) &= {}_2F_1(a, b; \alpha + 1|x) \\ f_2(x) &= x^{-\alpha}\ {}_2F_1(a - \alpha, b - \alpha; 1 - \alpha|x). \end{aligned}$$

2.3. Applying a stochastic time change. Using the CIR process as in [**ACCZ03**] and [**AC03**], it was found that the martingale process constructed in analogy to the above procedure could not be calibrated to accurately match actual credit processes. In particular, the kurtosis of the distributions involved was too small. This problem was remedied by performing a stochastic time change. We take the same approach here, by applying a stochastic time change of the variance-gamma type. Then, the new kernel can be calculated as:

$$\bar{u}_\tau(y_0, y) = \int_0^\infty \tilde{u}_t(y_0, y)\tilde{\gamma}(t, \tau) dt$$

where $\tilde{\gamma}$ is the probability density function of the gamma distribution and is defined by:

$$\tilde{\gamma}(t, \tau) = \frac{t^{\tau/\nu - 1} e^{-t/\nu}}{\Gamma(\tau/\nu)\nu^{\tau/\nu}}. \tag{2.8}$$

2.4. Risk-neutralization. The stochastic process described above is sufficient to model the real-world process. We apply a time-dependent drift in order to match interest rate spreads in the risk-neutral setting. The procedure in [**ACCZ03**] and [**AC03**] for the CIR process can be summarized as follows: (i) a default boundary was placed in y-space; (ii) a transformation was applied that kept the volatility invariant and mapped the default boundary to zero; (iii) the partial differential equation was solved numerically; (iv) the stochastic time change was applied to the probability kernel.

An analogous procedure can be applied in the present case with the Jacobi process. As the continuous case is not the focus of the present paper, and as the risk-neutralization in the discrete case is not directly analogous to the continuous case, we do not give the explicit procedure here.

3. Discrete case

While it was possible to calibrate the process in the CIR case, the calculations involved were very numerically intensive due to the need to perform two-dimensional quadrature. In [**ACCZ03**] and [**AC03**], we discretized the continuous case merely by approximating the transition probabilities with integrals over the intervals of the credit classes.

Rather, we can approximate the continuous process with a discrete, finite state process that is a birth-death process. That is, only nearest neighbor transitions are allowed, and the transitions are Poisson processes. The analogy to the continuous case is obvious: instead of intervals corresponding to credit classes, we have adjacent sets of nodes on the lattice corresponding to credit classes.

With the discretization carried out in this way, we find that it is possible to calculate the transition probabilities in analytically closed form, making the computations much more efficient.

3.1. Underlying process. Define the infinitesimal generator $\mathcal{L}_N$ of a stochastic process ξ_t on the discrete lattice $\Lambda = \{0, \ldots, N\}$ by the finite difference operator:

$$\mathcal{L}_N = D(\xi)\Delta + [B(\xi) - D(\xi)]\,\nabla_+ \tag{3.1}$$

where $\Delta f(\xi) = f(\xi+1) + f(\xi-1) - 2f(\xi)$ and $\nabla_+ f(\xi) = f(\xi+1) - f(\xi)$ and:

$$\begin{aligned} B(\xi) &= (N-\xi)(\xi+\alpha+1) \\ D(\xi) &= \xi(N+\beta+1-\xi) \qquad \xi \in \Lambda. \end{aligned}$$

It can be shown that the difference operator $\mathcal{L}_N$ converges to the differential operator $\mathcal{L}$ described in the previous section, for N large. Here, convergence has the meaning that for all $f \in C^2$ and for all x such that $xN \in \Lambda^0 = \{1, \ldots, N-1\}$,

$$\mathcal{L}f(x) - \mathcal{L}_N f_N(xN) = \mathcal{O}(1/N),$$

where $f_N(xN) = f(x)$.

The infinitesimal generator $\mathcal{L}_N$ can be represented by a tri-diagonal matrix:

$$\mathcal{L}_N(\xi_0,\xi) = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & & \\ c_1 & b_1 & a_1 & 0 & \cdots & \\ 0 & c_2 & b_2 & a_2 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & c_{N-1} & b_{N-1} & a_{N-1} \\ & & & & c_N & b_N \end{pmatrix} \quad (3.2)$$

where

$$\begin{aligned} a_\xi &= B(\xi), \quad \xi = 0,\dots,N-1 \\ b_\xi &= -B(\xi) - D(\xi), \quad \xi = 0,\dots,N \\ c_\xi &= D(\xi), \quad \xi = 1,\dots,N. \end{aligned}$$

Written in matrix form, the condition for $\mathcal{L}_N$ to be the infinitesimal generator of a process where probability is conserved is for all of the rows to sum to zero. While this clearly holds on the interior points, it is also true for the zero-th and N^{th} row since $D(0) = 0$ and $B(N) = 0$ respectively. However, for a stochastic process that would be appropriate to model credit risk, we require absorption at zero to emulate default. This can be accomplished by making a measure change and transformation in analogy to the continuous case.

3.2. Constructing a local martingale process. In analogy to the continuous case, one can specify a measure change and transformation for a discrete process to obtain another infinitesimal generator $\tilde{\mathcal{L}}_N$ that gives a stochastic process that is a local martingale. In addition, if we choose the measure change appropriately then there will be probability conservation at the upper boundary and probability absorption at the lower boundary, which is what is desired in a credit model with the possibility of defaults at 0. We describe here the procedure and utilization of the measure change and transformation.

Given a real number ρ, we seek two linearly independent functions $f_{1,N}$ and $f_{2,N}$ on the lattice Λ that satisfy the equation $\mathcal{L}_N f_{i,N} = \rho f_{i,N}$ on the interior of Λ. That is,

$$\mathcal{L}_N(\xi_0,\xi) f_{i,N}(\xi) = \rho f_{i,N}(\xi_0) \qquad \xi_0 = 1,\dots,N-1. \quad (3.3)$$

In particular, the difference equation (3.3) that f_1 and f_2 satisfy converge to the differential equation given in (2.3). If we choose two linearly independent sets of terminal points $\{f_{1,N}(N-1), f_{1,N}(N)\}$ and $\{f_{2,N}(N-1), f_{2,N}(N)\}$, then we can use the recurrence relation:

$$D(\xi)f_{i,N}(\xi-1) - [B(\xi)+D(\xi)]f_{i,N}(\xi) + B(\xi)f_{i,N}(\xi+1) = \rho f_{i,N}(\xi)$$

obtained from (3.1) to iterate $f_{1,N}$ and $f_{2,N}$ backwards to obtain the functions on all of Λ. Assume that two coefficients c_1 and c_2 can be chosen such that the function

$$g_N = c_1 f_{1,N} + c_2 f_{2,N}$$

is strictly positive on Λ. In addition, choose two coefficients c_3 and c_4 and define the transformation $\upsilon = \Upsilon(\xi)$ as:

$$\Upsilon = \frac{c_3 f_{1,N} + c_4 f_{2,N}}{g_N} - \frac{c_3 f_{1,N}(0) + c_4 f_{2,N}(0)}{g_N(0)}.$$