Linear Algebra

Basics, Practice, and Theory

Bernard Gelbaum

PREFACE

This book has three parts: **Basics, Practice, and Theory.**

The study of linear transformations between finite-dimensional vector spaces is carried out ultimately, both in practice and in theory, in terms of matrices. This is true because every finite-dimensional vector space V is, for some natural number n, essentially the set of all n-tuples of elements of a field K, e.g., real or complex numbers. Thus much of this book is concerned with the study of n-dimensional real space $\mathbf{R}^n$ or n-dimensional complex space $\mathbf{C}^n$ and linear transformations among and within them.

The identification of V with one of those spaces can be made in an infinite number of ways and, so viewed, V can be studied abstractly. The consequence is that a linear transformation T can be represented as a matrix in infinitely many ways and the search for a useful matrix representation of T is regarded as the search for a useful identification of V with $\mathbf{C}^n$, in effect a search for a useful basis for V. This approach to linear algebra is described in **Part III** but is *not* the one used in **Part I**.

In **Part I** the emphasis is on matrices themselves, and on their algebraic relations. The fundamentals of linear algebra are presented in the context of matrices as rectangular arrays and of vectors as special kinds of matrices, i.e., $n \times 1$ or $1 \times n$ matrices. The more abstract development in **Part III** serves, in particular, to illuminate the concrete discussion of **Part I**.

Part I. Basics.

Chapter 1. *Orientation.* Here the motivations for the study of linear algebra are given in terms of five problem classes.

Chapter 2. *The PROCESS* **(Simple elimination).** Several versions are given of the elimination method for analysing a system of finitely many equations in finitely many unknowns. The experience with this kind of analysis is made to lead naturally to the study of matrix algebra and the language of linear algebra. The notions of invertible matrices, rank, dimension, solvability of systems of equations, etc., are repeatedly brought back to their roots in the different versions of elimination.

Chapter 3. *Determinants* **(A direct approach).** The standard approaches to determinant theory are replaced by a simpler description, again based firmly in the elimination methods discussed in Chapter 2. Fortunately the simpler definition of a determinant is not merely equivalent to the standard definitions. It provides also for an easier proof of the basic multiplication theorem for determinants ($det(AB) = det(A)det(B)$) and for a direct and algorithmic method for the evaluation of a determinant.

Chapter 4. *Useful Forms For Matrices.* From the standpoint of matrix algebra and the functional calculus for matrices, there is given a motivated study of a SQUARE matrix and the possibilities for reducing it to more easily manipulable, e.g., diagonal and Jordan normal, forms. The presentation here, like the presentation of determinant theory, is novel and makes more accessible the whole topic of useful forms for matrices. The possibility of doing most of functional calculus with *polynomial* functions of a matrix is emphasized and illustrated.

Part II. Practice.

Chapter 5. *Applying Linear Algebra.* Applications of the developments in **Part I** are given to a wide variety of "real world" problems, particularly those sketched in **Chapter 1** but others as well. The computational aspects, considered today to be central in current studies of linear algebra, are discussed at some length. As well there are treatments of linear programming (including the Bland rules and the Karmarkar algorithm), 0-sum 2-person game theory, and Markov chains.

Part III. Theory.

Chapter 6. *General Vector Spaces.* The contents of **Part I** are given a broader setting via a reformulation in terms of abstract vector spaces and their endo-, epi-, homo-, iso-, and monomorphisms. Generalizations to infinite-dimensional spaces, in particular to Hilbert space, are indicated for the basic results derived in **Part I**. The limitations on these generalizations are exemplified.

Chapter 7. *Quadratic Forms.* Sylvester's Law of Inertia is proved. Definite quadratic forms are examined and characterized by determinantal criteria. Quadratic forms are applied to the study of extrema in multivariable calculus.

Chapter 8. *Miscellany.* The elements of vector space duality, of multilinear forms (with some reference to determinants), and of tensor algebra conclude the book.

Numerical **Exercises** and **Examples** abound throughout the text. Some of the simpler facts about linear algebra are given as **Exercises** in the body of the text.

References such as **[MW]** refer to items in the **Bibliography**.

There are occasional notes on etymologies, particularly of those words that are peculiar and intrinsic to the subject of linear algebra.

Thanks are extended to Roberto Leggiero, who, at a crucial time for the writer, was a consultant at the SUNY/Buffalo University Computing Services. Mr. Leggiero's patience and assistance in showing the author how to use TEX for typesetting the contents of this book were most helpful.

SUGGESTIONS FOR COURSE OUTLINES

A one-semester introductory course can be based on **Sections 1.1 - 3.1** and **4.1 - 4.4**. The level of the course determines the thoroughness with which the material is covered. For example, the treatment in **Section 3.1** provides a simple, direct, and easily understood definition of the determinant function.

The study and derivation of the properties of **determinants** can be passed over lightly so long as the **WORKING RULES** are explained.

A second semester permits coverage of **Sections 4.5 - 4.8** and **Part II**.

In a third semester the material in **Section 3.1** and in **Part III** provides the foundation for the study of abstract vector spaces and their elaborations.

Note on terminology. In the mathematical community, the words "Gauss," "Jordan," "elimination,", "reduction," "row-reduction," and "echelon form" in appropriate combinations are used to describe the procedures and results arising in the analysis of systems of linear equations. The essential ideas are embodied in what is here called the *PROCESS*. Successively refined variants of the *PROCESS* are dubbed the GEM (Gauss Elimination Method) that yields the echelon form, the GJM (Gauss Jordan Method) that produces via extended row-reduction an echelon form in which each pivot is 1 and each pivot column is a canonical basis vector.

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PART I *BASICS*

CHAPTER 1

ORIENTATION

1.1. Where linear algebra is useful and used

The subject of linear algebra crops up in modern applied mathematics and in many ways. Linear algebra is used to address and solve practical problems for which the methods of differential and integral calculus are of little value. It also provides the means for getting concrete numerical answers to a vast array of problems that arise in the applications of calculus itself, e.g., in ordinary and partial differential equations, in integral equations, and, most spectacularly, in quantum mechanics. In the last instance certain infinite-dimensional generalizations of finite-dimensional linear algebra seem to provide the most successful mathematical and practical solutions to the problems in atomic physics (quantum mechanics).

In the next few paragraphs there are posed some very simple and yet very practical problems. Linear algebra plays a central role in the solution of each of these problems. They are readily grouped into classes characterized by the kinds of *matrix manipulations* used to solve the problems.

The problem classes are listed below. They are illustrated in this section. In later parts of the book there are thorough treatments of the most effective methods of linear algebra that can be used to solve the problems in a given class.

The mathematics — linear algebra — stands on its own feet. It is sometimes motivated and even to some extent assisted by an understanding of the technical details of the physics, biology, ecology, economics, etc., from which the problems in the various classes come. However to understand the mathematics of linear algebra one need not be at all familiar with any of applied fields mentioned.

THE PROBLEM CLASSES

I. Systems of finitely many linear equations in finitely many unknowns.

II. Stability of populations of living organisms, of mechanical systems, and of biological and chemical processes, etc.

III. Approximation of solutions to equations in classical analysis.

IV. Linear programming and game theory.

V. Functional equations and functional calculus.

There follow illustrative examples for each of the classes listed above.

The problem classes I and IV.

In the field of economics much use is made of so-called "input-output" matrices or arrays. The next **Example** in the logistics of nutrition and food supply gives the basic idea.

Example 1.1.1. Suppose that the nutrients: PROTEINS (P), FATS (F), and CARBOHYDRATES (C) for a population are to be derived from the eating of MEAT PRODUCTS (M), DAIRY PRODUCTS (D), and GRAIN PRODUCTS (G). Suppose further that the following table represents the proportions of the nutrients that the three kinds of foods provide. Slanted capital letters such as P, F, ..., G are symbols representing *both* the items PROTEINS, FATS, ..., GRAIN PRODUCTS *and* the amounts of these items as well. Similar remarks apply later as the example is elaborated. (Simple numbers have been chosen so as not to obscure the essential structure. Since only their proportions are of consequence these are chosen to be reasonable in what follows. Units of measurement are not identified.)

	M	D	G
P	7	1	2
F	4	1	0
C	0	2	23

Table 1.1.1.

The array of numbers in **Table 1.1.1** above is the input-output *matrix* relating the "inputs" MEAT PRODUCTS, DAIRY PRODUCTS, and GRAIN PRODUCTS to the "outputs" PROTEINS, FATS, and CARBOHYDRATES. Thus the input-output matrix

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 7 & 1 & 2 \\ 4 & 1 & 0 \\ 0 & 2 & 23 \end{pmatrix}$$

is the array of coefficients in the left members of the following system of equations:

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$$\begin{aligned} 7M + D + 2G &= P \\ 4M + D + 0G &= F \\ 0M + 2D + 23G &= C. \end{aligned} \tag{1.1.1}$$

These equations show how the inputs are combined to produce the desired outputs.

The horizontal strips of numbers in a matrix are its *rows*. The vertical strips of numbers are its *columns*. The numbers are called the *entries*. The *entry* in the ith row and the jth column is called the (i, j) entry. Thus in $\mathcal{A}$ the (2,1) entry is 4, the (3,3) entry is 23. A matrix having m *rows* and n *columns* is called an $m \times n$ matrix and its *size* is $m \times n$. Thus $\mathcal{A}$ is a 3×3 matrix of size 3×3.

> [*Matrix* is Latin for — womb, pregnant animal — and is derived from *mater* — mother. The Indo-European root is *ma* — mother — and appears in — mama, mamma, mammal, maieutic, etc. *Matrix* occurs in many forms of discourse in which the word is used to describe a mold or a medium that *holds* something or a group of things in place, e.g., the womb holding the fetus, the mold for casting type faces in printing, etc.
>
> The use of *matrix* in linear algebra stems, perhaps, from the way in which matrices are depicted, namely as arrays of numbers bracketed by parentheses that hold or contain them.]

Suppose next that MEAT PRODUCTS, DAIRY PRODUCTS, and GRAIN PRODUCTS may be supplied from CATTLE (B), SHEEP (S), HOGS (H), RICE (R), WHEAT (W), and CORN (K).

(To avoid a collision between "C" for "CARBOHYDRATES" and "C" for "CATTLE" or "C" for "CORN," "B" serves for "CATTLE" (Latin "BOS" for "CATTLE" or English "BEEF") and "K" serves for "CORN" ("KERNEL").

	B	S	H	R	W	K
M	2	1	3	0	0	0
D	10	1	0	0	0	2
G	0	0	0	4	6	10

Table 1.1.2.

The array of numbers in **Table 1.1.2** is the 3×6 input-output matrix

$$\mathcal{B} \stackrel{\text{def}}{=} \begin{pmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 6 & 10 \end{pmatrix}$$

relating the "inputs" CATTLE, SHEEP, HOGS, RICE, WHEAT, and CORN, to the "outputs" MEAT PRODUCTS, DAIRY PRODUCTS, and GRAIN PRODUCTS.

If specific numbers are given for the demands P, F, and C then solving the first system (1 1 1) for the three unknowns M, D, and G reveals the amounts of MEAT PRODUCTS, DAIRY PRODUCTS, and GRAIN PRODUCTS needed to satisfy the demands. The resulting values for M, D, and G can then be entered into the system of equations derived from Table 1.1.2 relating M, D, and G to B, S, H, R, W, and K, namely

$$\begin{aligned} 2B + 1S + 3H + 0R + 0W + 0K &= M \\ 10B + 1S + 0H + 0R + 0W + 2K &= D \\ 0B + 0S + 0H + 4R + 6W + 10K &= G. \end{aligned} \tag{1.1.2}$$

Replacing M, D, and G in (1.1.1) by the expressions in (1.1.2) leads to a third system, namely

$$\begin{aligned} 24B + 8S + 21H + 8R + 12W + 22K &= P \\ 18B + 5S + 12H + 0R + 0W + 2K &= F \\ 20B + 2S + 0H + 92R + 138W + 234K &= C. \end{aligned} \tag{1.1.3}$$

The coefficients in the left members (1.1.3) form the 3×6 input-output matrix

$$C \stackrel{\text{def}}{=} \begin{pmatrix} 24 & 8 & 21 & 8 & 12 & 22 \\ 18 & 5 & 12 & 0 & 0 & 2 \\ 20 & 2 & 0 & 92 & 138 & 234 \end{pmatrix}$$

relating the raw materials CATTLE, SHEEP, HOGS, RICE, WHEAT, and CORN to the desired nutrients PROTEINS, FATS, and CARBOHYDRATES.

The coefficient 24 of B in the first equation of (1.1.3)) is the (1,1) entry in the matrix above. How that coefficient is calculated exemplifies how all the coefficients in (1.1.3) are found:

$$\begin{aligned} P &= 7M + D + 2G \\ &= 7(2B + 1S + 3H + 0R + 0W + 0K) + 1(10B + 1S + 0H + 0R + 0W + 2K) \\ &\qquad + 2(0B + 0S + 0H + 4R + 6W + 10K) \\ &= (7 \cdot 2 + 1 \cdot 10 + 2 \cdot 0)B + \cdots \\ &= 24B + \cdots. \end{aligned} \tag{1.1.4}$$

The coefficients, 7, 1, and 2, in the formula preceding B in (1.1.4) come from *row* 1 of A. The coefficients, 2, 10, 0, come from *column* 1 of of B. So that *row* 1 of A combined in a special way with *column* 1 of B provides the (1,1) entry of C:

$$\begin{bmatrix} 7 \times 2 \\ + \\ 1 \times 10 \\ + \\ 2 \times 0 \end{bmatrix} = 24.$$

Row 1 of A is rotated 90° clockwise, its entries are matched against the entries in *column* 1 of B, the matched entries are multiplied together, and the results are added together.

A careful examination of the way in which C above is derived from A and B shows that the illustration above is a model for all the calculations of the entries in C.

The array of coefficients in the left members of (1.1.3) is related to the arrays in the left members of (1.1.1) and (1.1.2) in a manner that is the genesis of the idea for *multiplying* matrices. In other words

$$\underset{A}{\begin{pmatrix} 7 & 1 & 2 \\ 4 & 1 & 0 \\ 0 & 2 & 23 \end{pmatrix}} \underset{B}{\begin{pmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 6 & 10 \end{pmatrix}} = \underset{C}{\begin{pmatrix} 24 & 8 & 21 & 8 & 12 & 22 \\ 18 & 5 & 12 & 0 & 0 & 2 \\ 20 & 2 & 092 & 138 & 234 & \end{pmatrix}}$$

$$AB = C.$$

Note that the 3×6 matrix C arises from the particular combination of the 3×3 matrix A and the 3×6 matrix B. Note also that 3, the number of *columns* of A, is the same as 3, the number of *rows* of B.

More generally if

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nq} \end{pmatrix}$$

then A is an $m \times n$ matrix, B is a $n \times q$ matrix, and their *matrix product* or simply *product* AB *in the order just written* is defined to be the $m \times q$ matrix

$$P \stackrel{\text{def}}{=} \begin{pmatrix} p_{11} & \cdots & p_{1q} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mq} \end{pmatrix}.$$

in which the (i, j) entry p_{ij} in AB is given by the formula:

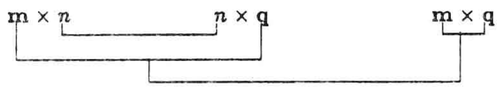
$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

In tighter form the preceding formula may be written:

$$p_{ij} = \sum_{\mathbf{K}=1}^{n} a_{i\mathbf{K}}b_{\mathbf{K}j}, \quad 1 \le i \le m, \quad 1 \le j \le q.$$

The match-up of the indices as shown above is helpful in remembering the formula. The index of summation (**K**) is emphasised in boldface. The equation above says that the entry in the *ith row* and *jth column* of the matrix AB arises by summing the products of the matched corresponding elements of the *ith row of A* and the *jth column* of B. Thus, with **boldface** letters marking the relevant entries, and with other entries suppressed, the formula above can be visualized as

$$\begin{pmatrix} \vdots & \ddots & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{in}} \\ \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} \cdots & \mathbf{b_{1j}} & \cdots \\ \ddots & \vdots & \ddots \\ \cdots & \mathbf{b_{nj}} & \cdots \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots \\ \cdots & \mathbf{p_{ij}} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$



The ith row of A is rotated 90° clockwise, aligned with the jth column of B so that corresponding entries match up, the matched entries are multiplied together and the products so formed are added together. The result is the (i, j) entry in AB:

$$\begin{bmatrix} \mathbf{a_{i1}} \times \mathbf{b_{1j}} \\ + \\ \mathbf{a_{i2}} \times \mathbf{b_{2j}} \\ + \\ \vdots \\ + \\ \mathbf{a_{in}} \times \mathbf{b_{nj}} \end{bmatrix} = \mathbf{p_{ij}}.$$

The matching of the entries in the jth column of B with the entries in the ith row of A reflects the need that the number n of *columns* of A be the same as the number n of *rows* of B. No restrictions are placed on the number of rows of A or on the number of columns of B.

Only because n, the *number of columns of* A, is the same as n, the *number of rows of* B, can the product AB as defined be formed. Conversely, if the number of columns of A is the same as the number of rows of B, then the product AB as defined can be formed.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix then AB as defined can be formed if and only if ("iff") $n = p$. The matrices A and B are called *compatible for multiplication* iff $n = p$. Their product AB is an $m \times q$ matrix.

Two different populations in widely separated parts of the world may have different agricultural techniques, different breeds of farm animals, different kinds of soil, etc. In that event the numbers in array **Table 1.1.2** might well be different for the second population, e.g.,

	B	S	H	R	W	K
M	1	2	5	0	0	0
D	6	2	0	0	0	5
G	0	0	0	8	5	6

Table 1.1.3.

If these two populations provide equal amounts of B, S, H, R, W, and K then the total production of the two populations of M, D, and G is governed by the array below:

	B	S	H	R	W	K
M	3	3	8	0	0	0
D	16	3	0	0	0	7
G	0	0	0	12	11	16

Table 1.1.4.

in which the entry in each box is the *sum* of the entries in the counterpart boxes of **Table 1.1.2** and **Table 1.1.3**.

The array in **Table 1.1.4** might be regarded as the *sum* of the arrays **Table 1.1.2** and **Table 1.1.3**. The idea of *adding* entries in corresponding boxes of two matched arrays is the genesis of the idea of *addition* of matrices. In mathematical terms addition of matrices may be illustrated as follows:

$$\underset{\mathcal{B}}{\begin{pmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 6 & 10 \end{pmatrix}} + \underset{\mathcal{D}}{\begin{pmatrix} 1 & 2 & 5 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 8 & 5 & 6 \end{pmatrix}}$$

$$= \underset{\mathcal{G}}{\begin{pmatrix} 3 & 3 & 8 & 0 & 0 & 0 \\ 16 & 3 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 12 & 11 & 16 \end{pmatrix}}$$

$$\mathcal{B} + \mathcal{D} = \mathcal{G}.$$

Each entry in $\mathcal{G}$ is the sum of the corresponding entries in $\mathcal{B}$ and $\mathcal{D}$. Two matrices A and B may be added together iff they are of the same *size*, e.g., each matrix should be a 3×6 matrix, like the matrices $\mathcal{B}$, $\mathcal{D}$. The sum of two matrices of equal size is another matrix of the same size, e.g., $\mathcal{G}$ is, like $\mathcal{B}$ and $\mathcal{D}$, a 3×6 matrix. Two matrices of the same size are called ***compatible for addition***.

The term "compatible" alone is to be interpreted in the context where it is used, e.g., in matrix multiplication or in matrix addition.

Matrix addition and matrix multiplication are the basic operations of matrix algebra. In the preceding survey and examples these operations crop up naturally in the course of organizing the analysis of the question of providing nutrition.

THE ELEMENTS OF MATRIX ALGEBRA

The following paragraphs constitute a short excursion into *matrix algebra*, the subject of combining matrices in various ways. For the most part each *entry* a_{pq} in the matrices to be considered is a *complex number*. Thus there is a symbol i such that $i^2 = -1 = (-i)^2$ and there are two real numbers r and s such that

$$a_{pq} \stackrel{\text{def}}{=} r + si \in \mathbf{C},\ r, s \in \mathbf{R}$$

When geometric interpretations of linear algebra are desirable each entry is taken to be real. Following the excursion the discussion of the applications resumes.

In formal terms the addition of two matrices can be described by the following equation:

$$\begin{pmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{m1} & . & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & . & b_{1n} \\ . & . & . \\ b_{m1} & . & b_{mn} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} a_{11}+b_{11} & . & a_{1n}+b_{1n} \\ . & . & . \\ a_{m1}+b_{m1} & . & a_{mn}+b_{mn} \end{pmatrix}.$$

Thus if the matrix

$$\begin{pmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{m1} & . & a_{mn} \end{pmatrix}$$

is denoted $(a_{ij})_{i,j=1}^{m,n}$ then the formula for the (r, c) entry of

$$(c_{ij})_{i,j=1}^{m,n} \stackrel{\text{def}}{=} (a_{ij})_{i,j=1}^{m,n} + (b_{ij})_{i,j=1}^{m,n}$$

is

$$c_{rc} = a_{rc} + b_{rc}.$$

It is natural to write

$$(a_{ij})_{i,j=1}^{m,n} + (a_{ij})_{i,j=1}^{m,n} \stackrel{\text{def}}{=} 2(a_{ij})_{i,j=1}^{m,n} \stackrel{\text{def}}{=} (2a_{ij})_{i,j=1}^{m,n}$$

and more generally, for any number t, $t(a_{ij})_{i,j=1}^{m,n} \stackrel{\text{def}}{=} (ta_{ij})_{i,j=1}^{m,n}$.

Exercise 1.1.1. If the two populations provide not *equal* amounts of B, S, H, R, W, and K but amounts in the ratio 2 : 3, what is the counterpart of **Table 1.1.4**?

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