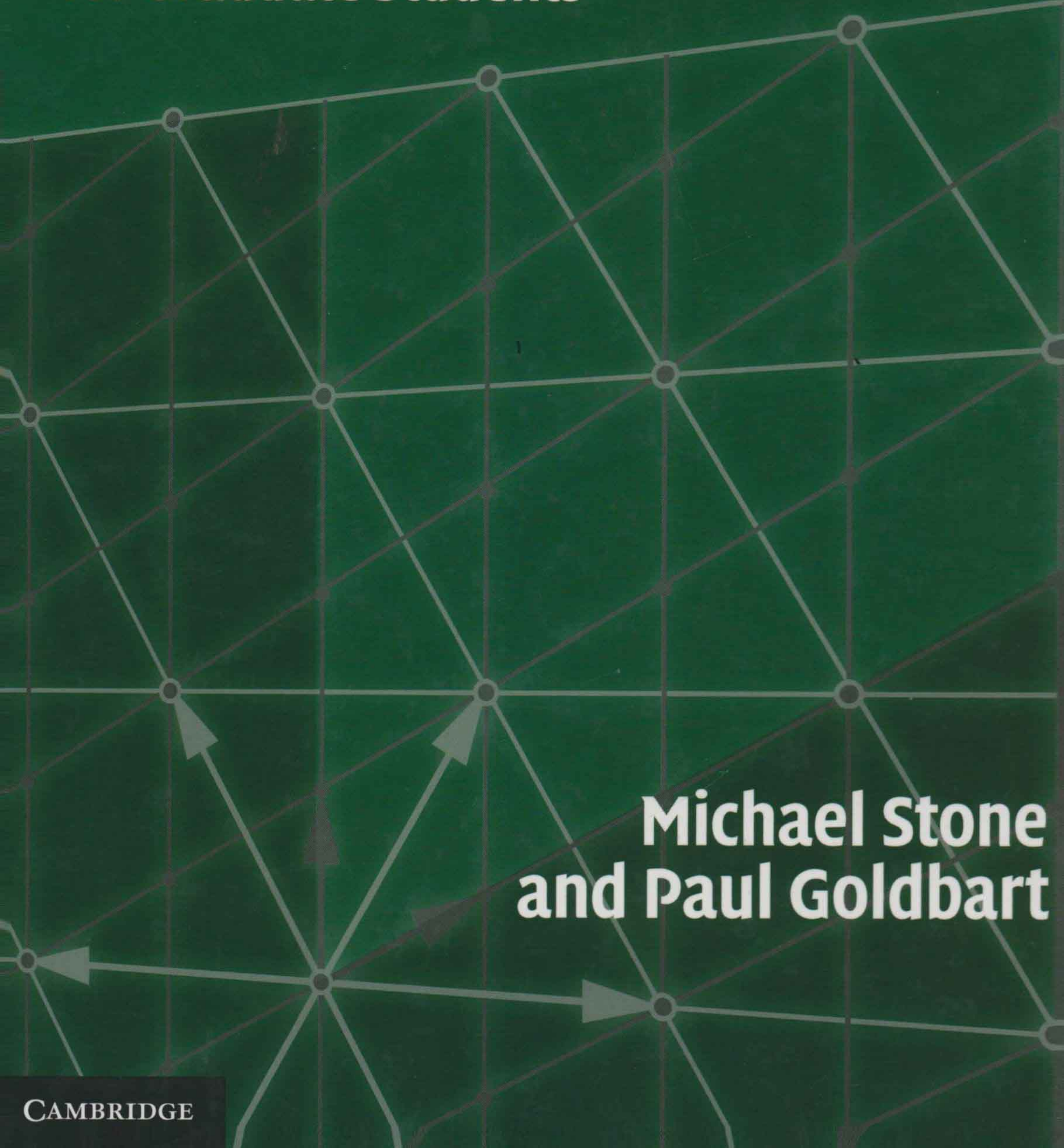
# Mathematics for Physics

## A Guided Tour for Graduate Students



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CAMBRIDGE

# MATHEMATICS FOR PHYSICS

## A Guided Tour for Graduate Students

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**Mathematics for Physics**

A Guided Tour for Graduate Students

An engagingly written account of mathematical tools and ideas, this book provides a graduate-level introduction to the mathematics used in research in physics.

The first half of the book focuses on the traditional mathematical methods of physics: differential and integral equations, Fourier series and the calculus of variations. The second half contains an introduction to more advanced subjects, including differential geometry, topology and complex variables.

The authors' exposition avoids excess rigour whilst explaining subtle but important points often glossed over in more elementary texts. The topics are illustrated at every stage by carefully chosen examples, exercises and problems drawn from realistic physics settings. These make it useful both as a textbook in advanced courses and for self-study. Password-protected solutions to the exercises are available to instructors at www.cambridge.org/9780521854030.

MICHAEL STONE is a Professor in the Department of Physics at the University of Illinois at Urbana-Champaign. He has worked on quantum field theory, superconductivity, the quantum Hall effect and quantum computing.

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To the memory of Mike's mother, Aileen Stone: $9 \times 9 = 81$.

To Paul's mother and father, Carole and Colin Goldbart.

# Preface

This book is based on a two-semester sequence of courses taught to incoming graduate students at the University of Illinois at Urbana-Champaign, primarily physics students but also some from other branches of the physical sciences. The courses aim to introduce students to some of the mathematical methods and concepts that they will find useful in their research. We have sought to enliven the material by integrating the mathematics with its applications. We therefore provide illustrative examples and problems drawn from physics. Some of these illustrations are classical but many are small parts of contemporary research papers. In the text and at the end of each chapter we provide a collection of exercises and problems suitable for homework assignments. The former are straightforward applications of material presented in the text; the latter are intended to be interesting, and take rather more thought and time.

We devote the first, and longest, part (Chapters 1–9, and the first semester in the classroom) to traditional mathematical methods. We explore the analogy between linear operators acting on function spaces and matrices acting on finite-dimensional spaces, and use the operator language to provide a unified framework for working with ordinary differential equations, partial differential equations and integral equations. The mathematical prerequisites are a sound grasp of undergraduate calculus (including the vector calculus needed for electricity and magnetism courses), elementary linear algebra and competence at complex arithmetic. Fourier sums and integrals, as well as basic ordinary differential equation theory, receive a quick review, but it would help if the reader had some prior experience to build on. Contour integration is not required for this part of the book.

The second part (Chapters 10–14) focuses on modern differential geometry and topology, with an eye to its application to physics. The tools of calculus on manifolds, especially the exterior calculus, are introduced, and used to investigate classical mechanics, electromagnetism and non-abelian gauge fields. The language of homology and cohomology is introduced and is used to investigate the influence of the global topology of a manifold on the fields that live in it and on the solutions of differential equations that constrain these fields.

Chapters 15 and 16 introduce the theory of group representations and their applications to quantum mechanics. Both finite groups and Lie groups are explored.

The last part (Chapters 17–19) explores the theory of complex variables and its applications. Although much of the material is standard, we make use of the exterior calculus, and discuss rather more of the topological aspects of analytic functions than is customary.

A cursory reading of the Contents of the book will show that there is more material here than can be comfortably covered in two semesters. When using the book as the basis for lectures in the classroom, we have found it useful to tailor the presented material to the interests of our students.

# Acknowledgments

A great many people have encouraged us along the way:

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- The staff of the US National Science Foundation and the US Department of Energy, who have supported our research over the years.

Our sincere thanks to you all.

# Contents

# 1
# Calculus of variations

We begin our tour of useful mathematics with what is called the *calculus of variations*. Many physics problems can be formulated in the language of this calculus, and once they are there are useful tools to hand. In the text and associated exercises we will meet some of the equations whose solution will occupy us for much of our journey.

## 1.1 What is it good for?

The classical problems that motivated the creators of the calculus of variations include:

(i) *Dido's problem*: In Virgil's *Aeneid*, Queen Dido of Carthage must find the largest area that can be enclosed by a curve (a strip of bull's hide) of fixed length.
(ii) *Plateau's problem*: Find the surface of minimum area for a given set of bounding curves. A soap film on a wire frame will adopt this minimal-area configuration.
(iii) *Johann Bernoulli's brachistochrone*: A bead slides down a curve with fixed ends. Assuming that the total energy $\frac{1}{2}mv^2 + V(x)$ is constant, find the curve that gives the most rapid descent.
(iv) *Catenary*: Find the form of a hanging heavy chain of fixed length by minimizing its potential energy.

These problems all involve finding maxima or minima, and hence equating some sort of derivative to zero. In the next section we define this derivative, and show how to compute it.

## 1.2 Functionals

In variational problems we are provided with an expression $J[y]$ that "eats" whole functions $y(x)$ and returns a single number. Such objects are called *functionals* to distinguish them from ordinary functions. An ordinary function is a map $f : \mathbb{R} \to \mathbb{R}$. A functional $J$ is a map $J : C^\infty(\mathbb{R}) \to \mathbb{R}$ where $C^\infty(\mathbb{R})$ is the space of smooth (having derivatives of all orders) functions. To find the function $y(x)$ that maximizes or minimizes a given functional $J[y]$ we need to define, and evaluate, its *functional derivative*.

### 1.2.1 The functional derivative

We restrict ourselves to expressions of the form

$$J[y] = \int_{x_1}^{x_2} f(x, y, y', y'', \cdots y^{(n)})\, dx, \tag{1.1}$$

where $f$ depends on the value of $y(x)$ and only finitely many of its derivatives. Such functionals are said to be *local* in $x$.

Consider first a functional $J = \int f dx$ in which $f$ depends only $x$, $y$ and $y'$. Make a change $y(x) \to y(x) + \varepsilon\eta(x)$, where $\varepsilon$ is a (small) $x$-independent constant. The resultant change in $J$ is

$$\begin{aligned} J[y+\varepsilon\eta] - J[y] &= \int_{x_1}^{x_2} \left\{ f(x, y+\varepsilon\eta, y'+\varepsilon\eta') - f(x, y, y') \right\} dx \\ &= \int_{x_1}^{x_2} \left\{ \varepsilon\eta \frac{\partial f}{\partial y} + \varepsilon \frac{d\eta}{dx}\frac{\partial f}{\partial y'} + O(\varepsilon^2) \right\} dx \\ &= \left[ \varepsilon\eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} (\varepsilon\eta(x)) \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx \\ &\quad + O(\varepsilon^2). \end{aligned}$$

If $\eta(x_1) = \eta(x_2) = 0$, the variation $\delta y(x) \equiv \varepsilon\eta(x)$ in $y(x)$ is said to have "fixed endpoints". For such variations the integrated-out part $[\dots]_{x_1}^{x_2}$ vanishes. Defining $\delta J$ to be the $O(\varepsilon)$ part of $J[y+\varepsilon\eta] - J[y]$, we have

$$\begin{aligned} \delta J &= \int_{x_1}^{x_2} (\varepsilon\eta(x)) \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx \\ &= \int_{x_1}^{x_2} \delta y(x) \left( \frac{\delta J}{\delta y(x)} \right) dx. \end{aligned} \tag{1.2}$$

The function

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \tag{1.3}$$

is called the *functional* (or *Fréchet*) derivative of $J$ with respect to $y(x)$. We can think of it as a generalization of the partial derivative $\partial J/\partial y_i$, where the discrete subscript "$i$" on $y$ is replaced by a continuous label "$x$", and sums over $i$ are replaced by integrals over $x$:

$$\delta J = \sum_i \frac{\partial J}{\partial y_i}\delta y_i \to \int_{x_1}^{x_2} dx \left( \frac{\delta J}{\delta y(x)} \right) \delta y(x). \tag{1.4}$$

### 1.2.2 The Euler–Lagrange equation

Suppose that we have a differentiable function $J(y_1, y_2, \ldots, y_n)$ of $n$ variables and seek its *stationary points* – these being the locations at which $J$ has its maxima, minima and saddle points. At a stationary point $(y_1, y_2, \ldots, y_n)$ the variation

$$\delta J = \sum_{i=1}^{n} \frac{\partial J}{\partial y_i} \delta y_i \tag{1.5}$$

must be zero for all possible $\delta y_i$. The necessary and sufficient condition for this is that all partial derivatives $\partial J/\partial y_i$, $i = 1, \ldots, n$ be zero. By analogy, we expect that a functional $J[y]$ will be stationary under fixed-endpoint variations $y(x) \to y(x) + \delta y(x)$, when the functional derivative $\delta J/\delta y(x)$ vanishes for all $x$. In other words, when

$$\boxed{\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0, \quad x_1 < x < x_2.} \tag{1.6}$$

The condition (1.6) for $y(x)$ to be a stationary point is usually called the *Euler–Lagrange* equation.

That $\delta J/\delta y(x) \equiv 0$ is a *sufficient* condition for $\delta J$ to be zero is clear from its definition in (1.2). To see that it is a *necessary* condition we must appeal to the assumed smoothness of $y(x)$. Consider a function $y(x)$ at which $J[y]$ is stationary but where $\delta J/\delta y(x)$ is non-zero at some $x_0 \in [x_1, x_2]$. Because $f(y, y', x)$ is smooth, the functional derivative $\delta J/\delta y(x)$ is also a smooth function of $x$. Therefore, by continuity, it will have the same sign throughout some open interval containing $x_0$. By taking $\delta y(x) = \varepsilon\eta(x)$ to be zero outside this interval, and of one sign within it, we obtain a non-zero $\delta J$ – in contradiction to stationarity. In making this argument, we see why it was essential to integrate by parts so as to take the derivative off $\delta y$: when $y$ is fixed at the endpoints, we have $\int \delta y'\, dx = 0$, and so we cannot find a $\delta y'$ that is zero everywhere outside an interval and of one sign within it.

When the functional depends on more than one function $y$, then stationarity under all possible variations requires one equation

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \tag{1.7}$$

for each function $y_i(x)$.

If the function $f$ depends on higher derivatives, $y''$, $y^{(3)}$, etc., then we have to integrate by parts more times, and we end up with

$$0 = \frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y^{(3)}}\right) + \cdots. \tag{1.8}$$

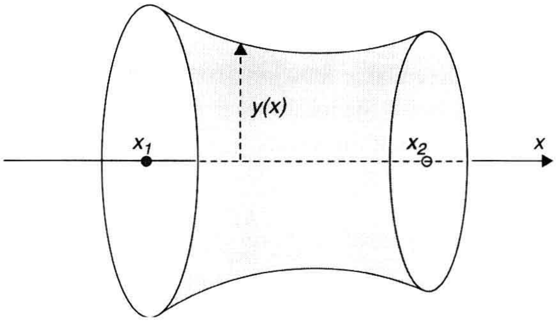


Figure 1.1 Soap film between two rings.

### *1.2.3 Some applications*

Now we use our new functional derivative to address some of the classic problems mentioned in the introduction.

*Example*: *Soap film supported by a pair of coaxial rings* (Figure 1.1). This is a simple case of Plateau's problem. The free energy of the soap film is equal to twice (once for each liquid–air interface) the surface tension $\sigma$ of the soap solution times the area of the film. The film can therefore minimize its free energy by minimizing its area, and the axial symmetry suggests that the minimal surface will be a surface of revolution about the $x$-axis. We therefore seek the profile $y(x)$ that makes the area

$$J[y] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx \tag{1.9}$$

of the surface of revolution the least among all such surfaces bounded by the circles of radii $y(x_1) = y_1$ and $y(x_2) = y_2$. Because a minimum is a stationary point, we seek candidates for the minimizing profile $y(x)$ by setting the functional derivative $\delta J/\delta y(x)$ to zero.

We begin by forming the partial derivatives

$$\frac{\partial f}{\partial y} = 4\pi\sqrt{1+y'^2}, \quad \frac{\partial f}{\partial y'} = \frac{4\pi y y'}{\sqrt{1+y'^2}} \tag{1.10}$$

and use them to write down the Euler–Lagrange equation

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0. \tag{1.11}$$

Performing the indicated derivative with respect to $x$ gives

$$\sqrt{1+y'^2} - \frac{(y')^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{y(y')^2y''}{(1+y'^2)^{3/2}} = 0. \tag{1.12}$$

After collecting terms, this simplifies to

$$\frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0. \tag{1.13}$$

The differential equation (1.13) still looks a trifle intimidating. To simplify further, we multiply by $y'$ to get

$$\begin{aligned} 0 &= \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} \\ &= \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right). \end{aligned} \tag{1.14}$$

The solution to the minimization problem therefore reduces to solving

$$\frac{y}{\sqrt{1+y'^2}} = \kappa, \tag{1.15}$$

where $\kappa$ is an as yet undetermined integration constant. Fortunately this nonlinear, first-order, differential equation is elementary. We recast it as

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{\kappa^2} - 1} \tag{1.16}$$

and separate variables

$$\int dx = \int \frac{dy}{\sqrt{\frac{y^2}{\kappa^2} - 1}}. \tag{1.17}$$

We now make the natural substitution $y = \kappa \cosh t$, whence

$$\int dx = \kappa \int dt. \tag{1.18}$$

Thus we find that $x + a = \kappa t$, leading to

$$y = \kappa \cosh \frac{x+a}{\kappa}. \tag{1.19}$$