READINGS IN LINEAR PROGRAMMING VAJDA

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By

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Preface

LINEAR Programming is a mathematical computing technique which has been developed during the last decade; it has already been useful in a great variety of planning problems, and forms by now an important tool in the outfit of an " operational research " worker.

The chapters of this book contain examples of applications, the stress being throughout on the practical aspect, i.e. on arithmetic rather than on mathematics. No attempt has been made to make the problems dealt with completely realistic, because a clear outline is more helpful in tracing the basic ideas than are involved explanations and a long series of computations. However, the problems are representative of those that have either occurred in practice, or have been treated in the ever-expanding literature of this subject.

Three theoretical chapters (I, XIV and XIX) have been included to make the book self-contained. They do not replace expositions which a pure mathematician would like to see as other sources exist to cater for him. A few of the practical chapters also contain theoretical matter where needed.

Chapter II deals with one of the simplest Linear Programming tasks: the Transportation Problem. It can be solved by very simple routines, and the subsequent chapters (III-VI) show how other problems can be transformed into this type, and hence solved by the same simple procedures. Chapter VII considers another problem which can be solved by a simple routine: maximal flow through a network. Chapters VIII-XI reduce other problems (one of them the transportation problem itself) to cases of this type, and these chapters contain some of the most recent developments of Linear Programming technique. Chapter XII explains how a certain problem of nearly transportation type can be solved, without referring to the most general method.

Chapter XIII introduces a problem which requires the knowledge of a more general routine (outlined in Chapter XIV) and Chapters XV-XVIII discuss some more problems of this type. Finally, Chapters XX-XXIV are concerned with problems where a deeper understanding of the theory (outlined in Chapter XIX) makes it possible to extract more information from the solution than a plain answer would give. Chapter XXIV, in particular, is addressed to those who are familiar with the Theory of Games.

At the end of some chapters there are simple examples which allow the reader to test his knowledge in a practical way. References to the literature will help him to extend his reading, if he wishes.

Only very elementary mathematics is required for the understanding of the book. It is hoped that the variety of cases analysed will prove the usefulness of this new branch of applied mathematics, and that it will be equally attractive to beginners in the field of operational research as well as to more experienced managerial personnel.

I should like to thank Mr. E. M. L. Beale, who has himself contributed to research in this field, for many stimulating and helpful discussions.

I am grateful to the Admiralty for permission to publish this book.

S. V.

Contents

I

Introductory

LINEAR Programming (L.P.) deals with maximizing or minimizing a linear expression of variables, called the " objective function," while the variables satisfy given linear equations or inequalities, referred to as constraints.

If the constraints are equations, and if their number equals that of the variables, then there is, in general, only one solution and the objective function is irrelevant. A genuine L.P. problem arises only when there are more variables than equations.

It is convenient to consider as the standard form of a L.P. problem that where the constraints are inequalities, and where it is understood that the variables may not have negative values. The constraints can be transformed into equivalent equations, by adding further non-negative (" slack ") variables. For instance, if a constraint were $a_1x_1 + a_2x_2 \leqslant b$, then we can write instead $a_1x_1 + a_2x_2 + y = b$.

On the other hand, if we have linearly independent equations, we can convert them into inequalities by expressing as many variables as there are equations in terms of the remaining ones (this is possible unless the system of equations is contradictory) and then requiring that these expressions should be non-negative. Another, less elegant way of converting equations into inequalities consists in writing each equation twice, once with the $\leqslant$ sign and once with the $\geqslant$ sign instead of the $=$ sign.

If there are also variables the values of which are not restricted as to their sign, then they can be eliminated; such a procedure should, however, not be applied to sign-restricted variables, because the requirement that their values must not be negative could easily be lost sight of. Again, a less elegant way of converting variables not restricted in sign to variables that are so restricted consists in replacing the former by the difference of two variables of the latter type. It is then always possible to find solutions in which at least one out of a pair of such variables is zero.

Of course, a system of linear inequalities or equations in sign-restricted variables need not have any solution at all. For instance, the system

$$x_1 + x_2 + x_3 = 5, \quad x_1 + x_2 - x_4 = 9$$

demands that $x_1 + x_2$ be not larger than 5, and at the same time not smaller than 9, which is evidently impossible.

It can also happen that a system of constraints is consistent, but that no finite maximum (or minimum) of the objective function exists. For example, if we wish to maximize x_1 subject to $x_1 - x_2 = 3$, then x_1 is clearly not restricted by any upper bound.

Finally, we remark that the routines for maximizing and for minimizing are analogous. As a matter of fact, the first could be used for the second as well, since the maximum of an expression is simply the negative of the minimum of that expression with opposite sign.

The following definitions will be used—

A *solution* is any set of variables (not necessarily non-negative) satisfying the constraints. Whenever we mean the solution of the whole L.P. problem, we refer to the *final*, or *optimal*, solution. A solution consisting of only non-negative values of the variables is *feasible*, and one that does not contain more than m positive variables, the others being zero (m is the number of constraints) is a *basic* solution. It can be proved that if a system has feasible solutions, then it has also basic feasible solutions, and these are the ones we shall, as a rule, try to determine.

If a basic solution contains less than m variables with positive values, then we call it *degenerate*.

REFERENCES

The algebraic theory of L.P. is contained in VAJDA [92].* This contains also a more rigorous definition of a "basis."

* Numbers in brackets refer to the bibliography.

II

Transportation Problem (I)

In due course we shall introduce a general routine for the solution of L.P. problems (see Chapter XIV). However, in special cases we can sometimes apply simplified routines and we start with an example where this is the case.

Assume that in a small town a need suddenly arises for a number of relief buses. In particular, 3, 3, 4, and 5 buses are required at points *A*, *B*, *C*, and *D* respectively. These 15 buses must come from the garages G_1, G_2, and G_3 where 2, 6, and 7 buses are ready for such an emergency. The manager would like to distribute the buses in such a way that the total of bus-minutes from the garages to their destinations is as small as possible. The time in minutes taken to travel from each garage to each destination is given in the following table—

	To	*A*	*B*	*C*	*D*
From G_1		13	11	15	20
G_2		17	14	12	13
G_3		18	18	15	12

If we denote the number of buses to be sent from G_i to *A*, *B*, *C*, and *D* respectively by x_{i1}, x_{i2}, x_{i3}, and x_{i4}, then the requirements can be expressed as follows—

(*a*) The buses at the garages are distributed among the destinations—

$$x_{11} + x_{12} + x_{13} + x_{14} = 2$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 6$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 7$$

(*b*) The buses come to the destinations from various garages—

$$x_{11} + x_{21} + x_{31} = 3$$
$$x_{12} + x_{22} + x_{32} = 3$$
$$x_{13} + x_{23} + x_{33} = 4$$
$$x_{14} + x_{24} + x_{34} = 5.$$

The " cost " to be minimized is the total time taken measured in bus-minutes, i.e.

$$13x_{11} + 11x_{12} + 15x_{13} + 20x_{14} + \dots + 12x_{34}.$$

This is a very simple case of L.P. All the coefficients are unity, and the pattern of variables occurring in any single equation is a very special one. Of the 7 equations only 6 are linearly independent, and thus a basic feasible solution will consist of variables with not more than 6 positive values.

It is easy to find such a basic feasible solution. Consider the following table which will be filled in—

		Numbers at destinations			
		3	3	4	5
Numbers	2				
at	6				
sources	7				

To begin with, take the cell of shortest time, which is that of x_{12} with its row and column labelled 2 and 3. Because 2 is the smaller, enter 2 into this cell, thereby allocating the 2 buses from G_1. We need one more bus at *B*, and thus we are now concerned with the reduced scheme

	3	1	4	5
6				
7				

Dealing with this table as we did with the previous one, we enter the smaller of 4 and 6 into the cell of x_{23}. Proceeding in the same way, the entries in the table are eventually—

	3	3	4	5
2	.	2	.	.
6	1	1	4	.
7	2	.	.	5

The totals of rows and columns are, of course, those originally given. The total time of this scheme is 197 bus-minutes, though we do not yet know whether this is the best possible scheme, i.e. that producing the shortest total travelling time.

In order to investigate this point, consider any empty cell, e.g. that of x_{11}. If we entered there 1, say, then we must take 1 away from x_{12}; to balance again, we must add 1 to x_{22} and then subtract 1 from x_{21}. Performing these transfers, we change the total cost by $c_{11} - c_{12} + c_{22} - c_{21} = 13 - 11 + 14 - 17 = -1$, i.e. the total cost would be reduced. Here c_{ij} is the time taken to travel from G_i to the j^{th} destination. On the other hand, if we wanted a positive entry for x_{14}, the cost would be changed by $c_{14} - c_{34} + c_{31} - c_{21} + c_{22} - c_{12} = 2$, so that this would produce an increase.

In this way we could examine each empty cell and find out whether the scheme could be improved. This method was called the "stepping-stone method" by Charnes and Cooper **[11]**. However, it would be very awkward if we had to make such an investigation for each empty cell, finding every time a circuit that leads from the empty cell by horizontal and vertical steps using only filled-in cells, and that returns to the empty cell considered. A short-cut is provided by the "modified" method, which works as follows: on the margins of the table we enter fictitious costs c_i for every row and d_j for every column, such that $c_i + d_j = c_{ij}$ for all occupied cells. These fictitious costs can be constructed by making c_1, say, equal to 0 whereby the others are implicitly fixed. (These fictitious costs need not be non-negative.) Consider now the first transfer mentioned above. A transfer of one bus changes the total cost by $c_{11} - c_1 - d_2 + c_2 + d_2 - c_2 - d_1 = c_{11} - c_1 - d_1$. Hence, if the cost of the unoccupied cell implied by the fictitious cost $c_1 + d_1$ is larger than the true cost c_{11}, then such a rearrangement leads to an improvement. The fictitious costs indicate, in the same way, whether an improvement is indicated, even if the transfers involve more than four cells, as in our second example. This is so because each more complicated transfer can be made up of a succession of simple ones.

If the implied fictitious cost equals the true cost, then an entry could be made in that cell, but the resulting reallocation would leave the total cost unchanged.

For the procedure which we have outlined to work, the number of occupied cells must equal the number of independent equations in the transportation problem, i.e. $n + m - 1$, where n and m are the numbers of sources and destinations respectively. This can be achieved, if necessary, by entering appropriate multiples of a small number into unoccupied cells, and replacing this small number by zero in the final solution.

In our problem the fictitious costs (f.c.) of rows and columns, and the implied fictitious costs of the cells unoccupied in our first solution, are given by the following table—

					f.c.
	14	.	9	8	0
	.	.	.	11	3
	.	15	13	.	4
f.c.	14	11	9	8	

Comparison with the true costs shows that it is worth while making a transfer into x_{11} (true cost 13, fictitious cost 14) and to effect this we alternately add to and subtract from x_{11}, x_{21}, x_{22}, x_{12}. We cannot transfer more than 1, because otherwise x_{21} would become negative. We obtain thus the following new solution, with the new fictitious costs as indicated—

	3	3	4	5	f.c.
2	1	1	.	.	0
6	.	2	4	.	3
7	2	.	.	5	5
f.c.	13	11	9	7	

If we compare the implied fictitious costs in the empty cells with their true costs, we find that the former are everywhere the smaller, and hence we have obtained the final, cheapest, solution. The total travelling time involved is 196 bus-minutes.

We now introduce a measure of efficiency of any given transportation scheme. For this purpose we ask ourselves which would be the most expensive scheme to satisfy the requirements. We find—

	3	3	4	5	f.c.
2	.	.	.	2	0
6	3	.	.	3	−7
7	.	3	4	.	−6
f.c.	29	24	21	20	

Here, since we want to find the most expensive total cost, we have found the final solution when all the implied fictitious costs are higher than the true costs. This is the case in the present table, and the highest total cost is 244 bus-minutes.

The efficiency of any scheme between the cheapest and the most expensive can be defined as the ratio between the actual and the largest possible reduction from the most expensive scheme. For instance, our first solution gave 197 bus-minutes. Its efficiency was therefore $(244 - 197)/(244 - 196) = 47/48$. The cheapest scheme has efficiency 1 and the most expensive has, naturally enough, efficiency zero. (This concept would be inapplicable if any of the costs were infinite.)

If there is more material at all sources taken together than there is required at all the destinations—e.g. when there is a supply of buses at the garages which exceeds the number of those actually needed at *A*, *B*, *C*, and *D*, then we add another, fictitious, destination, to which all surplus of material is deemed to be directed, at zero cost. The buses earmarked for this dummy destination would, of course, actually not be moved at all from the garages.

REFERENCES

The method of this chapter is based on that of CHARNES and COOPER [11]. Another method, based on that of FORD and FULKERSON [43], will be explained in Chapter X.

III

Caterer Problem

THE solution of a transportation problem is so simple that it is useful to transform other problems into this type, if this is possible. A case where it can be done will now be studied.

> A caterer undertakes to organize garden parties for a week and decides that he will need a supply of new napkins, each of which costs 2s. He will buy as few as possible and rather send used napkins to a laundry for cleaning. He finds that a laundry provides a service for 5d. a piece, and returns them within four days. There is also a quicker service, at 8d. a piece, and the napkins are then returned within two days.
>
> Making his plans for the week, the caterer estimates that he will need, during the seven days, 130, 70, 60, 100, 80, 90, and 120 napkins respectively. How many should he buy, and how many should he send to the laundry, for slow or for quick service, so as to keep his total expenses as low as possible?

Dealing with this problem algebraically, we denote the price of a new item by a, that of slow laundering by b, and that of fast laundering by c. The numbers required on the consecutive days are denoted by r_i $(i = 1, \ldots, 7)$. We also denote the numbers of napkins bought on the *i-th* day by x_i, of those sent to the laundry for slow service by y_i, and of those sent for quick service by z_i. Finally, the numbers of napkins which have been used but not yet sent to the laundry on the *i-th* day are denoted by t_i.

The first requirement which we consider is that the napkins bought on a particular day, and those coming back from the laundry after cleaning, must add up to that day's requirement. Thus

$$x_i = r_i \ (i = 1, 2)$$
$$x_i + z_{i-2} = r_i \ (i = 3, 4)$$
$$x_i + z_{i-2} + y_{i-4} = r_i \ (i = 5, 6, 7).$$

Further, we note that the number of pieces used on a particular day equals the number of those sent to the laundry and of those remaining soiled. Hence

$$r_1 = y_1 + z_1 + t_1$$
$$r_i + t_{i-1} = y_i + z_i + t_i \ (i = 2, \ldots, 7).$$

The r_t are known, and so we have 14 equations for 28 unknowns. Only non-negative values of the latter have any realistic meaning. The first two equations are, of course, already solved.

The objective function to be minimized is that of the total cost, i.e.

$$C = a(x_1 + \ldots + x_7) + b(y_1 + \ldots + y_7) + c(z_1 + \ldots + z_7).$$

We shall solve this problem by transforming it into one of transportation type, and for this purpose we must define " sources " and " destinations." We introduce the following sources—

1. A store from which we buy new napkins.
2. The napkins sent to and returned from the laundry.

Our destinations are the requirements on the various days, and also a store of used napkins and of those bought in excess of requirements, if any.

The next task is to determine the marginal totals in the transportation table. The given requirements appear as totals of sources as well as of destinations. But we must still find the number of napkins to be bought. It does no harm if we imagine that we buy too many, provided we imagine that any excess requirement is put into the final inventory, at zero cost.

It is of interest, in this connection, to determine an upper limit to the number of those that might have to be bought. It is easy to see that, if it takes p days to recover the pieces from the slow service, then the highest number that might conceivably have to be bought is the largest of the totals of p successive requirements. In our present example this is 390, and this number would certainly have to be bought if there were no faster service in existence. It is also easy to see that the number to be bought is at least the largest of the totals of q successive daily requirements, if q is the number of days after which the fast service returns the napkins. This total is here 210.

We turn to the " costs " of the transportation model. If the source is the store of napkins to be bought, then the cost of any item from this source is a if the piece is used for one of the daily requirements, and 0 if it is not used, i.e. if it goes directly into the final inventory.

If the source is the pile of already used and then laundered napkins, then the cost is b for an item used after 4 or more days, and c if used after 2 or 3 days. (This assumes that we send the napkins to the fast service only if the slow service would not do.) If a piece is not used again, then it goes into the final inventory, and the cost of this is again zero. We have therefore the following table of costs, with the indication of the totals of sources and destinations on the margins—

	130	70	60	100	80	90	120	390
390	*a*	*a*	*a*	*a*	*a*	*a*	*a*	0
130			*c*	*c*	*b*	*b*	*b*	0
70				*c*	*c*	*b*	*b*	0
60					*c*	*c*	*b*	0
100						*c*	*c*	0
80							*c*	0
90								0
120								0

The first row refers to the napkins that must be bought, and the last column to the final inventory. The two totals are, of course, the same.

The cost table is not yet complete. The cells where we have not entered any cost must not be used at all, since we do not send napkins to the laundry before we have used them. In order to exclude these cells from the final solution, we assign to them a very high cost so that it becomes cheaper to leave these cells out of consideration, whatever the entries in the other cells.

We now start solving the problem in its transportation form. Consider, to begin with, the first row. The cheapest entry is in the last column; but we know already that at least 210 must actually be bought, and thus not more than 180 can go directly into the final inventory, represented by the last column. Thus we fill in the first row—

	130	70	60	100	80	90	120	390
390	130	70	10					180

Now we consider the columns in turn, remembering that there are cells which must remain empty. In each column we first fill the cheapest cells, at cost *b* per unit. Then we consider the cells with cost *c*. Here we enter the lowest first, because this means that we use for the laundries from the fast service as late a source as possible, thus giving earlier ones a chance of returning their napkins through the slower, cheaper service. This produces the arrangement—

	130	70	60	100	80	90	120	390
390	130	70	10					180
130			50	30	50			0
70				70				0
60					30		30	0
100						90	10	0
80							80	0
90								90
120								120

The lines drawn in the table indicate the limits of the various cost factors. It is easy to see (e.g. by the stepping-stone method) that as long as the number of napkins actually bought is fixed at 210, no improvement is possible while c exceeds b. Whether an improvement is possible by buying more napkins can be determined by subtracting from the top right entry, 180, and entering the balance in some other cell of the first row. Whether this is useful depends on the value of $(c - b)/(a - b)$. In the present case one finds that no improvement is possible. However, if the price of the fast service were, for instance, $c = 12$ per piece, then the following scheme would be cheaper—

	130	70	60	100	80	90	120	390
390	130	70	20					170
130			40	30	60			
70				70				
60					20		40	
100						90		10
80							80	
90								90
120								120

We must interpret the entries in terms of the original variables x_i, y_i, z_i, and t_i, since these are the ones which tell us what to do.