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INVERSE PROBLEMS AND NONLINEAR EVOLUTION EQUATIONS

SOLUTIONS, DARBOUX MATRICES AND WEYL–TITCHMARSH FUNCTIONS

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Inverse Problems and Nonlinear Evolution Equations

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To the memory of Dora Sakhnovich,
mother of one of the authors and grandmother of another,
with love and gratitude

Preface

This book is based on the method of operator identities and the related theory of S-nodes, both developed by L. A. Sakhnovich. The notion of the transfer matrix function generated by the S-node plays an essential role.

We represent fundamental solutions of various important systems of differential equations using the transfer matrix function, that is, either directly in the form of the transfer matrix function or via the representation in this form of the corresponding Darboux matrix, when Bäcklund–Darboux transformations and explicit solutions are considered. The transfer matrix function representation of the fundamental solution yields, in turn, a solution of an inverse problem, namely, the problem to recover the system from its Weyl function. Weyl theories of self-adjoint and skew-self-adjoint Dirac systems (including the case of rectangular matrix potentials), related canonical systems, discrete Dirac systems, a system auxiliary to the N-wave equation and a system rationally depending on the spectral parameter are obtained in this way.

The results mentioned above on Weyl theory are applied to the study of the initial-boundary value problems for integrable (nonlinear) wave equations via the inverse spectral transformation method. The evolution of the Weyl function is derived for many important nonlinear equations, and some uniqueness and global existence results are proved in detail using these evolution formulas.

Generalized Bäcklund–Darboux transformation (GBDT) is one of the main topics of the book. It is presented in the most general form (i.e. for the case of the linear system of differential equations depending rationally on the spectral parameter). Applications to the Weyl theory and various nonlinear equations are given. Recent results on explicit solutions of the time-dependent Schrödinger equation of dimension $k+1$ are formulated in order to demonstrate the possibility to apply GBDT to linear systems depending on several variables.

Pseudospectral and Weyl functions of the general-type canonical system are studied in detail in Appendix A.

The last Chapter 9 of the book contains formulations and solutions of the inverse and half-inverse *sliding* problems for radial Schrödinger and Dirac equations, including the case of Coulomb-type potentials. Those results appeared first in 2013.

The reading of the book requires some basic knowledge of linear algebra, calculus and operator theory from standard university courses. All the necessary definitions and results on the method of operator identities and some other additional material is presented in Chapter 1. Moreover, several classical theorems, which are important for the book (e.g. the first Liouville theorem, Phragmen–Lindelöf theorem, Montel's theorem on analytic functions) are formulated in Appendix E.

Chapter 9 was written by L. A. Sakhnovich. Appendix C was written jointly by A. L. Sakhnovich and L. A. Sakhnovich. Chapters 2 and 3 were written by A. L. Sakhnovich and I. Ya. Roitberg. The rest of the book was written by A. L. Sakhnovich.

The book contains results obtained during the last 20 years (or slightly more), and the idea of writing such a book was first considered many years ago. A. L. Sakhnovich is grateful to J. C. Bot for the enthusiastic discussions of the project. The authors are very grateful to F. Gesztesy for his initiation and support of the present (final) version of the book.

A great part of the material, which is presented in the book, appeared during the last 5–6 years as a result of research supported by the Austrian Science Fund (FWF) under grants no. Y330 and no. P24301, and by the German Research Foundation (DFG) under grant no. KI 760/3-1.

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Notation

A^*	operator adjoint to A
B	de Branges space, see (A.11)
B_1	subspace of B, $B_1 = \widehat{U}L_1$
$\mathfrak{B}$	space, see (A.168)
$\mathfrak{B}_1$	space, see (A.168)
$B(\mathbf{G}, \mathbf{H})$	set of bounded operators acting from **G** into **H**
$B(\Omega) = B^0(\Omega)$	class of bounded on Ω functions (or matrix functions)
$B^N(\Omega)$	class of N times differentiable functions or matrix functions f on the domain Ω such that $\sup \|f^{(N)}\| < \infty$
$\mathbb{C}$	complex plane
$\mathbb{C}_+$	open upper half-plane
$\mathbb{C}_-$	open lower half-plane
$\overline{\mathbb{C}}_\pm$	closed upper (lower) half-planes
$\mathbb{C}_M$	open half-plane $\Im z > M$
$\mathbb{C}_M^-$	open half-plane $\Im z < -M$
$C(\Omega)$	class of continuous on Ω functions (or matrix functions)
$C^N(\Omega)$	class of N times differentiable functions or matrix functions f on the domain Ω such that $f^{(N)}$ is continuous
col	column, $\mathrm{col}\begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$
diag	diagonal matrix
$\mathrm{diag}\,\{\alpha_1, \alpha_2\}$	diagonal matrix with the entries (or block entries) α_1 and α_2 on the main diagonal
EG	acronym for *explicitly generated*, a class of potentials $\{C_k\}$ determined by the *admissible triples*, see Notation 5.38
$\widetilde{\mathrm{EG}}$	a class of potentials $\{C_k\}$, see Notation 5.48
I	identity operator
I_p	$p \times p$ identity matrix
$\mathrm{Im}\, A$	image of the operator A
$\Im$	imaginary part of either complex number or matrix
$\mathrm{Ker}\, A$	kernel of the operator A
L_0	subspace, see (A.10)
L_1	subspace, see (A.9)
$L^2(H)$	space, see (A.4)

$L^2_p(d\tau)$	Hilbert space with the scalar product $\langle f, g\rangle_\tau = \int_{-\infty}^{\infty} g(z)^* d\tau(z) f(z)$
$\mathfrak{M}(D,\varphi)$	mapping of a GW-function into potential ζ, see Definition 4.7
$\mathfrak{M}(\varphi)$	mapping of a GW-function into potential v, see Definition 3.29
$\widetilde{\mathfrak{M}}(G,x)$	GBDT-type mapping of G into $\widetilde{G}$, see Notation 7.9
$\mathbb{N}_0$	set of nonnegative integer numbers
$\mathcal{N}(r,z)$ or $\mathcal{N}(r)$	set of values (at z) of Möbius transformations, see Notation 5.10
$\mathcal{N}_u(x,z)$	set of values (at z) of Möbius transformations, see Notation 2.13 for the self-adjoint and Notation 3.1 for the skew-self-adjoint cases
$\mathcal{N}(\mathfrak{A})$	set of functions (of Möbius transformations), see (0.8)
$\mathcal{N}(\mathfrak{A}_A)$	set of functions (of Möbius transformations), see Notation 1.25
$\mathcal{N}(n)$	set of functions (of Möbius transformations), see Notation 5.27
$\mathcal{N}(S,\Phi_1)$	set of Herglotz functions, see Notation 1.24
$\mathcal{N}(\mathcal{U})$	set of functions (of Möbius transformations), see Notation 1.46
$\mathbf{P}_M(j)$	class of matrix functions with property-j, see Section 3.1 and Notation 3.1
$\mathbb{R}$	real axis
R_α	operator, see (A.23)
$\mathfrak{R}$	real part of either complex number or matrix
$S_{m_2\times m_1}(\Omega)$	class of Schur matrix functions, see Notation 3.2
span	linear span
$\overline{\text{span}}$	closed linear span
Tr	operator (matrix, in particular) trace
$\hat{U}$	operator, see (A.5)
$\overline{z}$	complex conjugate to z
δ_{ij}	Kronecker delta
$\sigma(A)$	spectrum of an operator A
$[D, \mathcal{M}]$	commutator $D\mathcal{M} - \mathcal{M}D$
$(\cdot,\cdot)_{\mathbf{H}}$	scalar product in $\mathbf{H}$
$\bigwedge$	vector product
$\|\cdot\|$	l^2 vector norm or the induced matrix norm

Contents

0 Introduction

In recent years, the interplay between inverse spectral methods and gauge transformation techniques to solve nonlinear evolution equations has greatly benefited both areas. The notions of the Weyl and scattering functions, Möbius (linear-fractional) and Bäcklund–Darboux transformations, Darboux matrices and nonlinear integrable equations are all interrelated. The purpose of this book is to treat this interaction by actively using various aspects of the *method of operator identities* (and S-nodes theory, in particular). Thus, Weyl functions that have been initially introduced in the self-adjoint case also prove very useful for solving non-self-adjoint inverse problems and Goursat problems for nonlinear integrable equations. The Darboux matrix can be presented in the form of the transfer matrix function from the system theory, and the Bäcklund–Darboux transformation can be fruitfully applied in the multidimensional case of $k > 1$ space variables. (We write matrix function and vector function meaning matrix-valued function and vector-valued function, respectively.) Some simple examples (as well as several important results) can already be seen in Chapter 1, where basic definitions and statements to make the book self-contained are also presented.

The famous Schrödinger (Sturm–Liouville) equation is usually considered in the form

$$-\frac{d^2}{dx^2}y(x,z) + v(x)y(x,z) = zy(x,z). \tag{0.1}$$

A great number of fundamental notions and results of analysis has been first introduced and obtained for this equation. The list includes the Weyl disc and point, Weyl solution, spectral and Weyl (or Weyl–Titchmarsh) functions, Bäcklund–Darboux transformation, transformation operator and solutions of the inverse problems, and so on. One also has to mention its connections with the Lax pairs, the method of the inverse scattering transform, and nonlinear integrable equations. We can refer to the already classical books [39, 195, 205, 209, 213, 217, 323], though numerous new important papers, surveys and books appear regularly. Among the more recent developments are the notions of bispectrality and $\mathcal{PT}$-symmetry (see [93] and [41], respectively). It is of special interest that practically all of these notions are related in one or another way.

For each continuous $v(x)$ $(0 \le x < \infty)$ and $z \neq \overline{z}$, there is a so called Weyl solution y_w of the Schrödinger equation such that $\int_0^\infty |y_w(x,z)|^2 dx < \infty$ (see, for instance, p. 60 in [196]). Here, we denote by $\overline{z}$ the complex number conjugate to z. Let y_1 and y_2 satisfy the Schrödinger equation and initial conditions

$$y_1(0) = \sin c, \; y_1'(0) = -\cos c; \quad y_2(0) = \cos c, \; y_2'(0) = \sin c \quad \left(y' := \frac{d}{dx}y\right).$$

Then, y_w admits representation $y_w(x,z) = \varphi(z)y_1(x,z) + y_2(x,z)$. Function φ is called the Weyl or Weyl–Titchmarsh function and is extremely important in spectral theory. The Weyl–Titchmarsh approach can be developed for a much wider class of

Schrödinger equations, for matrix Schrödinger equations, and for various other important systems. For the case of the Hamiltonian systems, some basic results have been obtained by Hinton and Shaw [141, 142]. *Canonical system*

$$dy(x,z)/dx = \mathrm{i}zJH(x)y(x,z), \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad H(x) \geq 0 \tag{0.2}$$

with locally summable *Hamiltonian* H, is a particular case of the Hamiltonian systems and a classical object of analysis. Here, J and $H(x)$ are $m \times m$ matrices, I_p is the $p \times p$ identity matrix, $2p = m$, and inequality $H(x) \geq 0$ means that

$$H(x) = H(x)^*$$

($H(x)$ is self-adjoint) and the spectrum of $H(x)$ is nonnegative. (In general, inequality $S_1 \geq S_2$ means that $S_1 = S_1^*$, $S_2 = S_2^*$ and $S_1 - S_2 \geq 0$.) The summability of a matrix function means that its entries belong to L^1 (the entries are summable). The spectral theory of the general-type canonical systems is studied in Appendix A.

The study of the canonical system includes, in turn, such particular cases as the Schrödinger matrix equation (0.1), where *potential* v is a $p \times p$ matrix function, and the well-known self-adjoint Dirac-type (also called Dirac, Zakharov–Shabat or AKNS) system:

$$\frac{d}{dx}y(x,z) = \mathrm{i}(zj + jV(x))y(x,z), \tag{0.3}$$

$$j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}. \tag{0.4}$$

Usually, we consider systems (0.1)–(0.3) on the finite intervals $[0, l]$ or semiaxis $[0, \infty)$. One can find further references and results on these systems, for instance, in [24, 84, 85, 101, 118, 133, 137, 228, 234, 289, 290], and on the Weyl (Weyl–Titchmarsh or M-) functions for these systems in [118, 195, 205, 289, 290]. Transformations of the Dirac and Schrödinger systems into canonical are given in Subsections 1.1.1 and 1.1.2 of Chapter 1. We note that Dirac (Dirac-type) systems differ from the radial Dirac systems which appeared earlier (and were also called Dirac). Radial Dirac systems are discussed in detail in Chapter 9 (see also some results from Section 8.2).

We recall that fundamental solutions of the first order differential systems are square nondegenerate matrix functions, which satisfy these systems (and generate in an apparent way all other solutions). The fundamental $m \times m$ solution of the canonical system (0.2) is normalized by the condition

$$W(0,z) = I_m. \tag{0.5}$$

Parameter matrix functions $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ of order p, such that

$$\mathcal{P}_1(z)^*\mathcal{P}_1(z) + \mathcal{P}_2(z)^*\mathcal{P}_2(z) > 0, \quad \begin{bmatrix} \mathcal{P}_1(z)^* & \mathcal{P}_2(z)^* \end{bmatrix} J \begin{bmatrix} \mathcal{P}_1(z) \\ \mathcal{P}_2(z) \end{bmatrix} \geq 0 \tag{0.6}$$

play an essential role in our book. Next, follow three basic definitions [281, 289, 290].

Definition 0.1. A pair of $p \times p$ matrix functions $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$, meromorphic in the upper half-plane $\mathbb{C}_+$, is called *nonsingular with property-J* if the first inequality from (0.6) holds in one point (at least) of $\mathbb{C}_+$ and the second inequality from (0.6) holds in all the points of analyticity of $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ in $\mathbb{C}_+$.

Remark 0.2. It is apparent that if the first inequality from (0.6) holds in one point of $\mathbb{C}_+$, it holds everywhere, except, possibly, some set of isolated points.

Put

$$\mathfrak{A}(l,z) = \mathfrak{A}(z) = W(l,\overline{z})^* = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}. \tag{0.7}$$

Definition 0.3 ([290, p. 7]). Matrix functions $\varphi(z)$, which are obtained via the transformation

$$\varphi(z) = \mathrm{i}\,(a(z)\mathcal{P}_1(z) + b(z)\mathcal{P}_2(z))\,(c(z)\mathcal{P}_1(z) + d(z)\mathcal{P}_2(z))^{-1} \tag{0.8}$$

(where the pairs $\mathcal{P}_1(z)$, $\mathcal{P}_2(z)$ of "parameter" matrix functions are nonsingular with property-J, $\det(c\mathcal{P}_1 + d\mathcal{P}_2) \not\equiv 0$), are called Weyl functions of the canonical system on the interval $[0,l]$.

We denote the class of the Weyl functions of the canonical system on $[0, l]$ by the acronym $\mathcal{N}(\mathfrak{A})$ or simply by $\mathcal{N}(l)$. The discs (Weyl discs) $\mathcal{N}(l)$ are embedded in one another, that is, $\mathcal{N}(l_1) \subseteq \mathcal{N}(l_2)$ for $l_1 > l_2$. It is shown in Appendix A that under a rather weak "positivity" condition on H, we have $\bigcap_{l<\infty}\mathcal{N}(l) \neq \emptyset$, and matrix function $\varphi \in \bigcap_{l<\infty}\mathcal{N}(l)$ satisfies inequality

$$\int_0^\infty \begin{bmatrix} I_p & \mathrm{i}\varphi(z)^* \end{bmatrix} W(x,z)^* H(x) W(x,z) \begin{bmatrix} I_p \\ -\mathrm{i}\varphi(z) \end{bmatrix} dx < \infty \quad (z \in \mathbb{C}_+). \tag{0.9}$$

In other words, the entries of $W(x,z)\begin{bmatrix} I_p \\ -\mathrm{i}\varphi(z) \end{bmatrix}$ belong $L^2(0,\infty)$, which is similar to the definition of the Weyl function of the Schrödinger equation on $[0, \infty)$. Hence, the definition below.

Definition 0.4. Holomorphic functions φ such that inequality (0.9) holds are called Weyl functions of system (0.2) on $[0, \infty)$.

Most of the direct problems in this book consist of the description of the set of Weyl functions, construction of Weyl functions and the study of their existence and uniqueness, whereas inverse problems, which we consider here, usually deal with the recovery of systems from their Weyl functions. Direct and inverse problems are considered either for systems determined on the finite interval $[0, l]$ *or for systems on the semiaxis* $[0, \infty)$.

Though all the notions mentioned in the Introduction are important for our book, *the Weyl functions, Bäcklund–Darboux transformations and Darboux matrices are the* *principal ones*. Various explicit formulas in the book are obtained through a new version of the well known in the spectral theory and integrable nonlinear equations (see, for instance, [10, 29, 70–72, 83, 84, 140, 192, 199, 206, 209, 213, 215, 231, 335] and references therein) Bäcklund–Darboux transformation (BDT). The BDT transforms the initial equation or system into another one from the same class and also transforms solutions of the initial equation into solutions of the transformed one. Let us illustrate this by the oldest and most popular example, that is, by the Schrödinger equation (0.1), where the scalar potential v is real valued (i.e. $v = \overline{v}$) and $z = \overline{z}$. Assume that $h(x) = \overline{h}(x)$ satisfies (0.1), when $z = c$, that is, $-h'' + vh = ch$ ($h'' := \frac{d^2}{dx^2}h$). Then, one can rewrite (0.1) in the form

$$(\mathcal{A}^*\mathcal{A} + cI)\, y(x,z) = zy(x,z), \tag{0.10}$$

where I is the identity operator, and $\mathcal{A}$ and $\mathcal{A}^*$ are first order differential expressions:

$$\mathcal{A}f = \left(\frac{d}{dx} - \frac{h'}{h}\right) f, \quad \mathcal{A}^* f = -\left(\frac{d}{dx} + \frac{h'}{h}\right) f.$$

The transformed equation is given by the formula

$$(\mathcal{A}\mathcal{A}^* + cI)\, y(x,z) = zy(x,z). \tag{0.11}$$

It easy to see that (0.11) is again the Schrödinger equation, but potential v is transformed into $\widetilde{v} = v - 2(\frac{h'}{h})'$. Notice further that

$$(\mathcal{A}\mathcal{A}^* + (c-z)I)\, \mathcal{A} = \mathcal{A}\, (\mathcal{A}^*\mathcal{A} + (c-z)I)\, .$$

Hence, it follows that if $y(x,z)$ satisfies (0.10), then $\widetilde{y} := \mathcal{A}y$ satisfies (0.11). All solutions of the transformed equations can be constructed in this way. Under rather weak conditions, the spectra of operators A^*A and AA^* may differ only at zero, and so under certain conditions, the spectra of Schrödinger operators L and $\widetilde{L}$ associated with differential expressions $-\frac{d^2}{dx^2} + v$ and $-\frac{d^2}{dx^2} + \widetilde{v}$ may only differ at c. The Bäcklund–Darboux-type (and related commutation) methods of inserting and removing eigenvalues of Schrödinger operators go back historically to Jacobi, Bäcklund and Darboux [29, 84, 146] with decisive later contributions by Crum, Deift and Gesztesy [82, 87, 88, 117, 123]. (See a detailed account in Appendix G in [118].)

One can apply the BDT again to the already transformed equation (0.11) and so on (iterated BDT). There is also a somewhat more complicated binary BDT ([2, 209]). It proves that if v satisfies a nonlinear integrable equation, then $\widetilde{v}$ often satisfies it too, and so the BDT is used to construct solutions of the nonlinear equations.

Elementary Bäcklund–Darboux transformations for Dirac-type and more general AKNS systems one can find, for instance, in [66, 172]. Given first order initial and transformed systems $u' = G(x,z)u$ and $\widetilde{u}' = \widetilde{G}(x,z)\widetilde{u}$, their solutions are connected via the so called Darboux matrix w such that $w' = \widetilde{G}w - wG$. Clearly, if u satisfies the initial system $u' = Gu$, then wu satisfies the transformed one