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Rainer Picard, Des McGhee

PARTIAL DIFFERENTIAL EQUATIONS

A UNIFIED HILBERT SPACE APPROACH

EXPOSITIONS IN MATHEMATICS 55

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Rainer Picard
Des McGhee

Partial Differential Equations

A Unified Hilbert Space Approach

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Dedication

We dedicate this book to the memory of Cliff Bartlett (1923–2010), a mathematician, artist and singer and a member of the Department of Mathematics (as it then was), University of Strathclyde for over thirty years, retiring in the mid-1980s. Cliff offered great encouragement and support to both of us in the early parts of our careers, to DM as a newly appointed lecturer and to RP during a year at Strathclyde as a Visiting Lecturer.

Rainer Picard, Des McGhee

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Preface

Given that there is a multitude of well-written and useful textbooks and monographs on partial differential equations, one of the obvious concerns of prospective authors might be why there should be another one written. So, it is appropriate to explain at the outset some of the particular features which make this book different from other texts. It will be obvious to the reader that the present book is not a run-off-the-mill text book on linear partial differential equations. First of all, the whole approach – although (with some additional work) extendable to a more general Banach space setting – is established in a Hilbert space setting, as the title of this monograph indicates. Of course, a Banach space setting is more general and sometimes more appropriate, but usually the core results rely nevertheless on a Hilbert space solution theory, a fact sometimes only tacitly acknowledged. We hope to show that, for presenting core ideas, our focus on a Hilbert space setting is not a constraint, but rather a highly suitable approach for providing a more transparent and even fairly elementary framework for presenting the main issues in the discussion of a solution theory for partial differential equations.

The reader may also find many topics, dealt with elsewhere, presented here in a slightly different flavor. Indeed, the building blocks (such as extrapolation and interpolation spaces, sums of operators, vector-valued Laplace transform) are largely well known with some of the ideas dating back to the early 1960s, see e.g. [35], [26] for the idea of interpolation/extrapolation spaces. Therefore, it has been and still is surprising to us that the full power of these concepts, which we utilize in this approach, has not been previously exploited to the extent we have found so useful. The differences are somewhat subtle and a more superficial reader may fail to appreciate how different our perspective on the theory of linear partial differential equations is. Indeed, it is this perspective on our approach which may be considered the most innovative feature of this monograph. In contrast to many other books, which are either focusing on specific types of partial differential equations or on a collection of tools for solving a variety of problems associated with various specific linear partial differential equations, we are attempting to assume a more global point of view on the issues involved.

Our approach can be classified as a functional analytical one, but this says very little, since nowadays it is the accepted standard to employ functional analytic language to formulate PDE problems. It may, however, come as a surprise that a Hilbert space setting is sufficiently general to cover the core issues of solving PDE problems. We focus on the case of *linear* partial differential equations, which is of course in one way a severe constraint, but given that by a rule of thumb non-linear problems, if they are at all well-posed, are frequently solvable by using a priori estimates and a fixed point argument based on perturbations of the linear theory, we see the restriction to linear partial differential equations more as foundation laying rather than an exclusion of non-linear issues.

A natural guideline for approaching problem solving is provided by Hadamard's celebrated criteria for well-posedness:[1]

- Uniqueness: there is at most one solution,
- Existence: there is at least one solution (at least for a dense set of data),
- Continuous Dependence: the dependence of the solution on the data is locally (or weakly locally) uniformly continuous.

Compared to these fundamental requirements, qualitative properties of a solution are a secondary consideration. This remark applies in particular to the issue of regularity in connection with solving linear partial differential equations.

The historical focus on regularity issues has fostered a number of guiding ideas, which in the light of our approach appear as occasionally misleading. Among these are the notions that

- partial differential equations are best classified according to the regularizing properties of their solution operators,
- one type of space should be used for solving "all" problems associated with partial differential equations,
- problems involving elliptic partial differential equations are "easier" than those involving parabolic partial differential equations, which are again "easier" than those involving hyperbolic partial differential equations.

The systematic approach presented here will shed a different and hopefully more illuminating light on these and other issues by

- proposing a different classification scheme,
- advocating the construction of "tailor-made" distribution spaces adapted to the particular equation at hand,

[1] These requirements are straightforwardly illustrated, if we consider the "problem" of finding a solution of the equation $F(x) = f$, where F denotes a mapping between – say – metric spaces. The properties are that F must be injective, with dense range and with F^{-1} being (at least weakly) locally uniformly continuous. We note in particular that the (weakly) local uniform continuity of F^{-1} allows the extension of F to its closure $\overline{F}$ given by $\overline{F}(x) := \lim F(y)$ for any sequence y converging to x such that $F(y)$ also converges. Indeed, for two sequences $y^{(k)}$, $k = 0, 1$, converging to x such that $F(y^{(k)})$, $k = 0, 1$, are also convergent, we have that $F(y^{(0)})$, $F(y^{(1)})$ must converge to the same limit, so that $\overline{F}$ is well-defined, i.e. F is closable. Weakly local uniform continuity is indeed characterized by mapping Cauchy sequences to Cauchy sequences.

- showing that from our point of view parabolic and hyperbolic partial differential equations are, in a way, "easier" than elliptic partial differential equations.

It will also be seen that our general framework is sufficiently powerful to be applied to general evolution equations. However, to flesh out the ideas presented, we shall illustrate by examples and consider particular systems of partial differential equations from mathematical physics.

Accordingly, the book is divided into several natural parts. In Chapter 1 we supply some additional material on functional analysis[2] in Hilbert space which may be difficult to find elsewhere. Chapter 2 introduces the idea of what we shall call Sobolev lattices. In Chapter 3, as a first application, we consider partial differential equations with constant coefficients in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$. The results are extended to tempered distributions (set in a suitably extended Sobolev lattice). Then in Chapter 4 the ideas presented are transferred to a more general framework covering a large class of abstract evolution equations. In Chapter 5 this general setting is exemplified by applications to a variety of initial-boundary value problems from mathematical physics. The concluding Chapter 6 offers a new approach to initial boundary value problems by expanding on the ideas and concepts presented.

The material is based on lecture notes developed for introductory and advanced graduate level courses on partial differential equations and functional analysis given by the first author over the past three decades at the Rheinische Friedrich-Wilhelms-Universität at Bonn, Germany, at the University of Wisconsin-Milwaukee, Wisconsin, USA, and at the Technische Universität Dresden, Germany, and on a series of lectures given at the Strathclyde University, Glasgow, Scotland, UK. This development from lecture notes has led to proofs being given in more detail than one might normally expect in a monograph. This may slow the readers progress, but it should, however, make the text not only useful as a resource for courses on the topic but also make it suitable as a text for a reading course or for self-study.

Apart from the novel approach the material presented in this monograph may in many ways be considered elementary, however, researchers will nevertheless find new results for particular evolutionary system from mathematical physics in later parts of this monograph as well as a very different perspective on seemingly familiar evolutionary problems.

This book has been in preparation for a number of years and many colleagues have contributed to the effort either directly through discussion and comment on research papers or at seminar or indirectly through general support and encouragement. The first of us (RP) would like to particularly acknowledge the hospitality of the Department and Mathematics and Statistics, University of Strathclyde, that he has enjoyed first as a young Visiting Researcher and more recently as a Visiting Professor, while DM would mention all colleagues, past and present, in the Department and Mathematics and Statistics, University of Strathclyde, but particularly Adam McBride, Wilson

[2] We have chosen to assume that the reader is familiar with basic functional analysis in Hilbert space.

Lamb and Mike Grinfeld of the Applied Analysis Group. Both of us would like to thank Rolf Leis and Gary Roach for encouragement and support throughout our careers.

Finally, we both wish to acknowledge the love and support of our wives, Brigitte and Alison, and families – their patience and understanding (of the effort required if not the mathematics itself!) has been necessary, and we hope sufficient, to see this task to a successful conclusion.

Dresden / Glasgow, January 2011 Rainer Picard, Des McGhee

Nomenclature

$\to$	mapping, as in $A \to B$
$\to$	strong convergence, as in $x_n \xrightarrow{n\to\infty} x_\infty$ for the convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ to its limit x_∞ in the norm topology
$\rightharpoonup$	weak convergence, as in $x_n \xrightharpoonup{n\to\infty} x_\infty$ for the weak convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ to its weak limit x_∞, i.e. in the weak topology
$\wedge$	logical "and"
$\vee$	logical (non-exclusive) "or"
$\bigwedge_{\ldots}$	for all ... / for every ...
$\bigvee_{\ldots}$	there is ... / there exists ...
$\mapsto$	maps to, as in $x \mapsto x^2$
$\bar{A}$	closure of A
A^*	adjoint relation or mapping to a relation or mapping A
$\mathring{A}$	closure of a closable operator A restricted to elements in $\mathring{C}_\infty(\Omega)$ for some open subset $\Omega \subseteq \mathbb{R}^{n+1}$, $n \in \mathbb{N}$
$\operatorname{cof}(A)$	co-factor matrix associated with a square matrix A
$\operatorname{cof}(A)^\top$	transposed co-factor matrix associated with a square matrix A, adjunct matrix
z^*	complex conjugate of a complex number z
$L(H, H)$	continuous linear mappings on H
$\mathring{C}_\infty(\Omega)$	space of smooth functions with compact support contained in the open subset $\Omega \subseteq \mathbb{R}^{n+1}$, $n \in \mathbb{N}$
∂	derivatives $(\partial_0, \partial_1, \ldots, \partial_n)$ in $\mathbb{R}^{1+n}$, $n \in \mathbb{N}$
$\widehat{\partial}$	spatial derivatives $(\partial_1, \partial_2, \ldots, \partial_n)$ in $\mathbb{R}^{1+n}$ with ∂_0 denoting the time derivative, $n \in \mathbb{N}$
∂_0	time derivative in $\mathbb{R}^{1+n}$, $n \in \mathbb{N}$
$\widehat{\partial}$	same as grad
$\widehat{\partial}\times$	same as curl
$\widehat{\partial}^\top$	same as div
$\oplus$	direct sum, orthogonal sum

$\bigoplus$	direct summation sign or orthogonal summation sign as in $\bigoplus_{t\in M} H_t$
Div	divergence of $(1,1)$-tensor fields
$\in$	element sign as in $x \in C$
$\in$	element function as in $x = \in(\{x\})$ giving the element of a set containing only one element
$\mathcal{L}_\nu$	Fourier–Laplace transform with parameter $\nu \in \mathbb{R}^{n+1}$
$\mathbb{L}_\nu$	temporal Fourier–Laplace transform with parameter $\nu \in \mathbb{R}$
$\widehat{f}(\cdot - \mathrm{i}\,\nu)$	same as $\mathcal{L}_\nu f$
Grad	symmetric part of the covariant derivative of a vector field
$H_{\nu,k}$	short for $H_k(\partial_\nu + \nu)$, $k \in \mathbb{Z}$
$H_{\nu,0,0}$	short for $H_{\nu,0} \otimes H$
$\mathbb{HL}$	Hardy–Lebesgue space
i	imaginary unit
$\langle \cdot \vert \cdot \rangle_X$	inner product of the inner product space X
$E^{-1/2}[X]$	inner product space derived from the inner product space X by modifying the inner product $\langle \cdot \vert \cdot \rangle_X$ to $\langle \cdot \vert E \cdot \rangle_X$, where $E : X \to X$ is continuous, linear, symmetric and strictly positive definite
$\lceil r \rceil$	smallest integer greater than or equal to the real number r (roof)
$\lfloor r \rfloor$	largest integer less than or equal to the real number r (integer part, floor)
$\bigcap$	big intersection symbol, as in $\bigcap M = \{y \vert \bigwedge_{X\in M} y \in X\}$ or $\bigcap_{X\in M} X$
Δ	spatial Laplacian, same as $\widehat{\partial}^2$ or $-\vert\widehat{\partial}\vert^2$
$\vert\widehat{\partial}\vert^2$	negative spatial Laplacian, same as $-\Delta$ or $-\widehat{\partial}^2$
$\widehat{\partial}^2$	spatial Laplacian, same as Δ or $-\vert\widehat{\partial}\vert^2$
ds_x	line element at a point x
ds	line element
Lin	linear hull, as in $\mathrm{Lin}_{\mathbb{K}} A$, the smallest linear space over the field $\mathbb{K}$ containing the set A
$-A$	the relation $\{(a,-b) \vert (a,b) \in A\}$ with $A \subseteq X \times Y$
$-[A]$	the set of all negatives of elements in the set A
$[\{0\}]f$	null space or kernel of a mapping or function f
$N(f)$	null space or kernel of a mapping or function f
$\mathbb{C}$	field or set of complex numbers
$\mathfrak{Re}$	real part

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$\mathfrak{Im}$	imaginary part
$\mathbb{K}$	field or set of numbers (either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$)
$\mathbb{N}$	monoid or set of natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{R}$	field or set of real numbers
$\mathbb{Z}$	group or set of integers
e	vector $(1, 1, \ldots, 1)$
$\|A\|$	operator norm of a linear operator A between normed linear spaces
$\|A\|_{X\to Y}$	operator norm of a linear operator $A : X \to Y$, X, Y normed linear spaces
$\perp$	orthogonal, as in $x \perp y$
$M^\perp$	ortho-complement of M,
$A[M]$	the post-set or image of M with respect to a mapping or function A
$A[M]$	the post-set or co-domain of M with respect to the binary relation A
$A[X]$	the post-set of X of a relation $A \subseteq X \times Y$, co-domain, range or image of the mapping or function A
$R(A)$	range of the mapping or function A
2^B	the power set of B
A^B	the set of (left-total) mappings from B into A
$[M]A$	the pre-image of M with respect to a mapping or function A
$[M]A$	the pre-set of M with respect to a binary relation A
$[Y]A$	the pre-set of a relation A, the pre-image of Y with respect to a mapping or function A or the domain of A
$D(A)$	the domain of a mapping or function A
$\times$	Cartesian product as in $X \times Y$ or Cartesian multiplication sign as in $\times_{s\in M} H_x$, where X, Y, M, H_s are sets, $t \in M$
$\times$	vector product in $\mathbb{R}^3$ as in $x \times y$, where $x, y \in \mathbb{R}^3$
P_C	orthogonal projector onto the closed subspace C
$\varrho(A)$	resolvent set of operator A
$\varrho_{-\infty}(A)$	Sobolev lattice resolvent set of A
$A\big\|_M$	A restricted to M for a mapping $A : D(A) \subseteq X \to Y$, i.e. the mapping $A\big\|_M : D(A) \cap M \subseteq X \to Y$ where $x \mapsto A(x)$
R_H	Riesz mapping, which unitarily maps H^* onto H
$\sigma(A)$	spectrum of operator A
$C\sigma(A)$	continuous spectrum of A

$P\sigma(A)$	point spectrum of A
$R\sigma(A)$	residual spectrum of A
$P\sigma_{-\infty}(A)$	Sobolev lattice point spectrum of A
$R\sigma_{-\infty}(A)$	Sobolev lattice residual spectrum of A
$\sigma_{-\infty}(A)$	Sobolev lattice spectrum of A
$C\sigma_{-\infty}(A)$	Sobolev lattice continuous spectrum of A
supp	support
supp_{ν_0}	support in direction ν_0
supp_0	temporal support
dS_x	surface element at x
dS	surface element
$\overset{a}{\otimes}$	algebraic tensor product
$\otimes$	tensor product
$\bigcup$	big union symbol, as in $\bigcup M = \{y \mid \bigvee_{X \in M} y \in X\}$ or $\bigcup_{X \in M} X$
grad	vector analytic differential operator grad, gradient
curl	vector analytic differential operator curl, curl
div	vector analytic differential operator div, divergence
dV_x	volume element at x
dV	volume element

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