Continuation & Bifurcations: Numerical Techniques & Applications

Continuation and Bifurcations: Numerical Techniques and Applications

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Preface

In September 1989, a NATO Advanced Research Workshop on "Continuation and Bifurcations : Numerical Techniques and Applications" was held at the Katholieke Universiteit Leuven, Belgium. Participants came from 10 countries in Europe and North America and were mainly from universities and research institutes. This proceedings volume contains 26 of the 38 papers which were presented at the meeting. Abstracts of most other contributions are also included.

The central theme of the workshop was the solution of parameter dependent nonlinear problems using numerical continuation. More specifically the aims can be stated as : to describe typical bifurcation problems in scientific, engineering and industrial problems ; to discuss current mathematical ideas and new developments in numerical analysis and numerical techniques and to describe and evaluate program packages and to discuss future needs with respect to software.

The interests of the participants extended over the complete spectrum of theory, numerical analysis, software and applications, and this spread is reflected both in the composition of this volume and in several of the papers. For example, there are contributions on the application of Centre Manifold and Liapunov-Schmidt theory to derive low dimensional systems which can be analysed by normal form theory for dynamical systems or singularity theory. On the numerical analysis front there are contributions, for example, on the computation of homoclinic and heteroclinic orbits, on the detection of Hopf bifurcations, on the computation of bifurcations in the presence of symmetry, on the calculation of rotating waves, and on the use of inertial manifolds in the bifurcation analysis of the Kuramoto-Sivashinski equation. Also included are descriptions of software packages for use on personal computers, and contributions on the use of symbolic manipulation codes. Several interesting applications are described, including separation in 3-D Navier Stokes flows, chaos in electrical circuits, the dynamics of passive optical systems and Marangoni convection in crystal growth.

The editors wish to thank the participants who made the workshop so successful, and the NATO Scientific Affairs Division, the National Science Foundation of Belgium (N.F.W.O.), the Ministry of Education of the Flemish Government and the K. U. Leuven for their generous sponsorship of the workshop. Finally, the editors acknowledge the opportunity given by Kluwer to publish the proceedings in the NATO ASI Series.

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BIFURCATION TO ROTATING WAVES FROM NON-TRIVIAL STEADY-STATES

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ABSTRACT. This paper considers bifurcation problems which are equivariant with respect to the rotations (or translations) r_α, $\alpha\in[0,2\pi)$ and the two reflections s_1 and s_2 which generate a group that we call $O^2(2)$. The rotational equivariance forces the linearisation of the problem to have a zero eigenvalue at every non-trivial steady-state solution. We show that when this zero eigenvalue has algebraic multiplicity two but geometric multiplicity one then bifurcation to rotating (or travelling) waves occurs, subject to a non-degeneracy condition. This result is obtained by reformulating the problem as a standard steady-state bifurcation in the presence of a reflectional symmetry. Finally, the generic form of bifurcation from rotating waves is considered.

1. Introduction

We consider bifurcation from a *non-trivial* branch of steady-state solutions to rotating (or travelling) wave solutions of the time-dependent nonlinear problem

$$\frac{dX}{dt} + g(X, \lambda) = 0\ , \quad t\geq 0 \tag{1.1}$$

where g is a C^2-mapping from $H\times\mathbb{R}$ into H, a Hilbert space with inner product $<\ ,\ >$. In our previous paper (Aston, Spence and Wu (1989)) we assume that g is O(2)-equivariant, that is,

$$\gamma g(X, \lambda) = g(\gamma X, \lambda) \quad \forall\gamma\in O(2),\ X\in H \tag{1.2}$$

where O(2) is the group generated by the rotations r_α, $\alpha\in[0, 2\pi)$ and a reflection s. One example of this arises when the steady-state problem $g(X, \lambda)=0$ represents a boundary value problem in one space dimension, say σ, with periodic boundary conditions. In this case, g is often equivariant with respect to r_α, which acts by rotation or translation as

$$r_\alpha X(\sigma, t) = X(\sigma+\alpha, t), \quad \alpha\in[0, 2\pi),$$

and *one* of the two reflections s_1 and s_2 which act by

$$s_1X(\sigma,t) = X(-\sigma,t)$$

$$s_2X(\sigma,t) = -X(-\sigma,t).$$

In this paper, we extend our previous results to the case when g is equivariant with respect to r_α and *both* reflections s_1 and s_2. This situation arises in reaction-diffusion equations when the

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nonlinear reaction function is odd in X.

Other authors have considered bifurcation from the *trivial* solution to rotating waves which arises as a Hopf bifurcation (see for example Iooss (1984)). However, there is an additional complication when considering bifurcation from a non-trivial branch of steady-state solutions, namely that $g_X(X, \lambda)$ is singular at every solution point (see Lemma 2.1). We shall show how this difficulty may be overcome by using a phase condition which enables us to reformulate the problem as a standard, steady-state bifurcation in the presence of a reflectional symmetry. We then show (see Theorem 3.3) that bifurcation to rotating waves occurs when $g_X(X,\lambda)$ has a zero eigenvalue with geometric multiplicity one and algebraic multiplicity two, subject to a non-degeneracy condition. This is in direct contrast to Hopf bifurcation where a complex conjugate pair of eigenvalues must cross the imaginary axis to produce a branch of time-periodic solutions.

No numerical results are presented here but we refer the reader to Aston *et al* (1989) where results for the Kuramoto-Sivashinsky equation are presented. Other numerical results for bifurcation to rotating waves from non-trivial steady-state solutions are given in Scovel, Kevrekidis and Nicolaenko (1988) and Kevrekidis, Nicolaenko and Scovel (1989).

Finally, we assume throughout that H is *finite-dimensional* although the extension to infinite dimensions described in Aston *et al* (1989) applies here also.

2. Preliminary Theory

We now set up the problem in a framework suitable for the bifurcation analysis of Section 3. We are interested in solutions of (1.1), where g is equivariant with respect to the rotations r_α, $\alpha\in[0, 2\pi)$ and the reflections s_1 and s_2 which satisfy the relations

$$s_i r_\alpha = r_{2\pi-\alpha} s_i \ , \qquad i=1,2 \tag{2.1a}$$

$$s_1 s_2 = s_2 s_1 \ . \tag{2.1b}$$

We define $s_{12}:=s_1 s_2$ and we refer to the group generated by r_α, s_1 and s_2 as $O^2(2)$. Let us now introduce our terminology. We call (x, λ) a *steady-state* solution of (1.1) if

$$g(x, \lambda) = 0 \tag{2.2}$$

and $(X(t), \lambda)$ a *rotating wave* solution of (1.1) if

$$X(t) = r_{ct} x \tag{2.3}$$

where $c\in\mathbb{R}$ is the velocity of the wave and $x\in H$ is independent of time. We define the following subspaces of H:

$$H_s = \{x\in H : s_1 x = x\}$$

$$H_a = \{x\in H : s_1 x = -x\}$$

$$H^\Sigma = \{x\in H : \sigma x = x, \ \ \forall\sigma\in\Sigma\}$$

where Σ is any subgroup of $O^2(2)$. We will refer to $x\in H^\Sigma$ as $\Sigma-$*symmetric*. We also define

$$r_\alpha' x := \frac{d}{d\alpha}(r_\alpha x) \ ,$$

$$A := r_0' \ .$$

Then r_α' is a bounded linear operator on H since the group action is smooth on finite-dimensional Hilbert spaces (Knapp (1986)).

We assume, without loss of generality (Golubitsky, Stewart and Schaeffer (1988, p31)), that the inner product < , > is $O^2(2)$-invariant, that is

$$<\gamma x, \gamma y> = <x, y> \quad \forall x,y\in H,\ \gamma\in O^2(2).$$

For the sake of simplicity, we assume that $H^{O^2(2)} = \{0\}$ so that (1.1) has the trivial, steady-state solution $x=0$ for all $\lambda\in\mathbf{R}$. Applying the theory outlined in Aston (1990) to bifurcation from the $O^2(2)$-symmetric trivial solution shows that steady-state, symmetry-breaking bifurcation can occur where the resulting primary branches of solutions are symmetric with respect to $r_{2\pi/n}$, s_1 and $s_2r_{\pi/n}$ for some $n\in\mathbf{Z}^+$ (by the Equivariant Branching Lemma). Use of the relations (2.1) gives $(s_{12}r_{\pi/n})^2 = r_{2\pi/n}$ and so this group is precisely the dihedral group $\tilde{D}_{2n}$ generated by the "rotation" $s_{12}r_{\pi/n}$ and the reflection s_1. The primary branches of solutions thus lie in $H^{\tilde{D}_{2n}}\times\mathbf{R}$. The main result of this paper (Theorem 3.3) is to show that bifurcation from such a primary branch of steady-state solutions to a branch of rotating wave solutions of the form (2.3) occurs when the s_1-symmetry is broken. This result can be proved in the context of steady-state bifurcation theory using the equation which we now derive.

It is easily shown, by differentiating (2.1a) with respect to α, that the linear operator A anti-commutes with both s_1 and s_2 and hence commutes with s_{12}. Similarly, A commutes with r_α. It follows from this and the equivariance of g that a rotating wave solution $(r_{ct}x,\ \lambda)$ of (1.1) satisfies the "steady-state" equation

$$\tilde{g}(x,\ c,\ \lambda) := g(x,\ \lambda) + cAx = 0\ , \tag{2.4}$$

and that $\tilde{g}$ is equivariant with respect to r_α, $\alpha\in[0,2\pi)$ and s_{12} only. Thus, if $(x,\ c,\ \lambda)$ is a solution of (2.4), then $(r_\alpha x,\ c,\ \lambda)$ is also a solution for all $\alpha\in[0,2\pi)$ due to the r_α-equivariance of $\tilde{g}$, giving rise to an *orbit* of *conjugate solutions*. In order to eliminate this nonuniqueness, we introduce a phase condition of the form

$$<l,\ x> = 0, \quad l\in H. \tag{2.5}$$

We then rewrite (2.4) and (2.5) as

$$G(y,\lambda)=0,\quad G\colon Y\times\mathbf{R}\to Y,$$
$$G(y,\lambda):=\begin{bmatrix} g(x,\lambda)+cAx \\ <l,\ x> \end{bmatrix}, \tag{2.6}$$
$$y=(x,c)\in Y:=H\times\mathbf{R}.$$

and define an inner product on Y by

$$y_1\cdot y_2 = <x_1,\ x_2> + c_1c_2$$

where $y_i=(x_i,\ c_i)$, $i=1,2$. (See Aston *et al* (1989) for details of the analysis using a more general phase condition.)

Differentiating the r_α-equivariance condition for $\tilde{g}$ with respect to α leads to the following result.

Lemma 2.1

If $(x,\ c,\ \lambda)$ is a solution of (2.4), then

$$\tilde{g}_x(x,\ c,\ \lambda)Ax = 0\ . \tag{2.7}$$

When considering bifurcation from the trivial solution $x=0$, this result provides no information since $Ax=0$. However, since we are considering bifurcation from a non-trivial branch of (steady-state) solutions, $Ax\neq0$ and so $\tilde{g}_x(x, 0, \lambda)=g_x(x, \lambda)$ has a non-trivial null space at every non-trivial steady-state solution of (1.1). Also, since we are considering a branch of solutions with $x\in H^{\tilde{D}_{2n}}$, then $\mathrm{Null}(g_x(x, \lambda))$ must be $\tilde{D}_{2n}$-invariant (see Aston (1990)) and so $\gamma Ax\in\mathrm{Null}(g_x(x, \lambda))$ for all $\gamma\in\tilde{D}_{2n}$. However, since s_1 anti-commutes with A, we have $s_1Ax=-As_1x=-Ax$. Similarly, $s_{12}r_{\pi/n}Ax=Ax$. Thus, the one-dimensional space spanned by Ax is $\tilde{D}_{2n}$-invariant and so we obtain no new null-vectors. The phase condition (2.5) can be viewed as a means of eliminating this singularity in the system (2.6). In particular, if (x, c, λ) is a solution of (2.4), then $(Ax, 0)$ will not be in the null space of $G_y((x, c), \lambda)$ provided that the non-degeneracy condition

$$<l, Ax> \neq 0 \tag{2.8}$$

is satisfied.

Finally, the system (2.6) inherits certain equivariance properties from $\tilde{g}$. If we define group actions on Y in terms of those on H by

$$S_1y := (s_1x, -c),$$
$$S_2y := (s_2x, -c),$$
$$R_{\pi/n}y := (r_{\pi/n}x, c),$$

where $y=(x, c)\in Y$, then we have the following result which is proved using the invariance of the inner product.

Lemma 2.2

If $l\in H^{Z_{2n}}\cap H_\sigma$, where Z_{2n} is the cyclic group generated by $r_{\pi/n}$, then G is equivariant with respect to S_1 and $S_{12}R_{\pi/n}$ where $S_{12}:=S_1S_2$.

Note that solutions of (2.6) which satisfy $S_1y=y$ must have $c=0$ and automatically satisfy the phase condition and so they consist of steady-state solutions of (1.1) contained in H_s. Thus, the primary branch solutions referred to earlier consist precisely of $\tilde{D}_{2n}$-symmetric solutions of (2.6) where $\tilde{D}_{2n}$ is the group generated by S_1 and $S_{12}R_{\pi/n}$. Note that $S_{12}R_{\pi/n}y = (s_{12}r_{\pi/n}x, c)$ and so $S_{12}R_{\pi/n}$-symmetric solutions do not necessarily have $c=0$.

3. Analysis of bifurcation to rotating waves

We now apply standard symmetry-breaking bifurcation techniques to $G(y, \lambda)=0$ defined by (2.6) to prove the existence of a branch of rotating waves bifurcating from a non-trivial branch of steady-state solutions contained in $H^{\tilde{D}_{2n}}\times\mathrm{R}$ for some $n\in Z^+$, subject to a non-degeneracy condition.

The first step is to restrict the problem to an appropriate fixed point subspace of Y. Since the linear operator A commutes with r_α, $\alpha\in[0, 2\pi)$ and anti-commutes with s_1 and s_2, it follows that if $x\in H^{\tilde{D}_{2n}}$, then the one-dimensional subspace of H spanned by Ax is $\tilde{D}_{2n}$-invariant and so it must also be $\tilde{D}_{2n}$-irreducible since it has no proper subspaces. Thus, Ax is contained in one of the isotypic components of H and so only the corresponding "block" of $g_x(x, \lambda)$ will be singular at every steady-state solution $(x, \lambda)\in H^{\tilde{D}_{2n}}\times\mathbb{R}$ of (2.1) (see Aston (1990)). If $x\in H^{\tilde{D}_{2n}}$, then the irreducible representation of $\tilde{D}_{2n}$ on the subspace spanned by Ax is

$$s_1 = [-1]\ , \quad s_{12}r_{\pi/n} = [1]\ ,$$

and so the corresponding isotypic component is $H^{\tilde{Z}_{2n}}\cap H_a$ where $\tilde{Z}_{2n}$ is the cyclic group of order $2n$ generated by $s_{12}r_{\pi/n}$. As we are only interested in bifurcation associated with this isotypic component, we take our setting to be

$$\tilde{Y} := Y^{\tilde{Z}_{2n}} = H^{\tilde{Z}_{2n}}\times\mathbb{R} \tag{3.1a}$$

and restrict G accordingly. Thus, we consider the system

$$G(y,\lambda)=0, \quad G:\tilde{Y}\times\mathbb{R}\to\tilde{Y},$$
$$G(y,\lambda) := \begin{bmatrix} g(x,\lambda)+cAx \\ <l,x> \end{bmatrix}, \tag{3.1b}$$
$$y=(x,c)\in\tilde{Y}, \quad l\in H^{\tilde{Z}_{2n}}\cap H_{aa}.$$

Since $s_{12}r_{\pi/n}$ acts as the identity on $H^{\tilde{Z}_{2n}}$, the only non-trivial action of $\tilde{D}_{2n}$ on $\tilde{Y}$ is the reflection S_1. Using this reflection, we decompose $\tilde{Y}$ as $\tilde{Y} = \tilde{Y}_s\oplus\tilde{Y}_a$ where

$$\tilde{Y}_s := \{y\in\tilde{Y} : S_1 y = y\} = H^{\tilde{D}_{2n}}\times\{0\}$$

$$\tilde{Y}_a := \{y\in\tilde{Y} : S_1 y = -y\} = (H^{\tilde{Z}_{2n}}\cap H_a)\times\mathbb{R}\ .$$

Clearly, solutions of (3.1) with $y\in\tilde{Y}_s$ are steady-state, $\tilde{D}_{2n}$-symmetric solutions of (1.1). Since $\tilde{Y}_s$ and $\tilde{Y}_a$ are invariant under $G_y(y_s, \lambda)$ for all $y_s\in\tilde{Y}_s$, we denote the restrictions of $G_y(y_s, \lambda)$ to $\tilde{Y}_s$ and $\tilde{Y}_a$ by $G_y^s(y_s, \lambda)$ and $G_y^a(y_s, \lambda)$ respectively.

The analysis now follows similar lines to that of Aston *et al* (1989) and so we only summarise the main results briefly without proofs. Thus, we now assume that there exists a steady-state solution $(x_0, \lambda_0)\in H^{\tilde{D}_{2n}}\times\mathbb{R}$ of (2.1). We have already seen that at such a point $Ax_0\in\mathrm{Null}(g_x^0)$, where we use the notation $g_x^0 := g_x(x_0, \lambda_0)$, and so we now make the further assumption that the zero eigenvalue of g_x^0 has geometric multiplicity *one* and algebraic multiplicity *two*. This assumption can be summarised by the following conditions, where * denotes the adjoint operator :

$$\mathrm{Null}(g_x^0) = \mathrm{span}\{Ax_0\} \tag{3.2a}$$

$$\mathrm{Null}((g_x^0)^*) = \mathrm{span}\{\psi_1\} \tag{3.2b}$$

$$<\psi_1, Ax_0> = 0 \tag{3.2c}$$

$$<\psi_1, \zeta_1> \neq 0 \tag{3.2d}$$

where

$$g_x^0\zeta_1 + Ax_0 = 0, \quad <l, \zeta_1> = 0\ . \tag{3.2e}$$

Finally, we also assume that $l\in H^{\tilde{Z}_{2n}}\cap H_a$ is chosen so that

$$<l, Ax_0> \neq 0\ . \tag{3.2f}$$

Now symmetry-breaking bifurcation can occur if $G_y^a(y_0, \lambda_0)$ has a non-trivial null-space, where $y_0=(x_0, 0)$.