INTRODUCTORY COLLEGE MATHEMATICS

HACKWORTH
and
HOWLAND

SAUNDERS SERIES IN MODULAR MATHEMATICS

Indirect Measurement

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PREFACE

Indirect Measurement

This book is one of the sixteen content modules in the Saunders Series in Modular Mathematics. The modules can be divided into three levels, the first of which requires only a working knowledge of arithmetic. The second level needs some elementary skills of algebra and the third level, knowledge comparable to the first two levels. *Indirect Measurement* is in level 3. The groupings according to difficulty are shown below.

Level 1	Level 2	Level 3
Tables and Graphs	*Numeration*	*Real Number System*
Consumer Mathematics	*Metric Measure*	*History of Real Numbers*
Algebra 1	*Probability*	*Indirect Measurement*
Sets and Logic	*Statistics*	*Algebra 2*
Geometry	*Geometric Measures*	*Computers*
		Linear Programming

The modules have been class tested in a variety of situations: large and small discussion groups, lecture classes, and in individualized study programs. The emphasis of all modules is upon ideas and concepts.

Indirect Measurement is appropriate for all non-science students especially education and liberal arts majors. Its content is also essential for technical majors. In any case, it is well suited for freshman and sophomore students.

Indirect Measurement begins by emphasizing skill in using angle and triangle notation while presenting the concept and uses of similar triangles. Then the module solves indirect measurement problems using proportions from similar triangles as a way of introducing trigonometric ratios. Also included is the use of the Pythagorean Theorem to find the missing side of a right triangle.

In preparing each module we have been greatly aided by the valuable suggestions of the following excellent reviewers: William Andrews, Triton College, Ken Goldstein, Miami-Dade Community College, Don Hostetler, Mesa Community College, Karl Klee, Queensboro Community College, Pamela Matthews, Chabot College, Robert Nowlan, Southern Connecticut State College, Ken Seydel, Skyline College, Ara Sullenberger, Tarrant County Junior College, and Ruth Wing, Palm Beach Junior College. We thank them, and the staff at W. B. Saunders Company for their support.

Robert D. Hackworth
Joseph W. Howland

NOTE TO THE STUDENT

OBJECTIVES:

Upon completing this unit the reader is expected to demonstrate the following skills and concepts:

1. to be able to use angle and triangle notation correctly.

2. to be able to identify similar triangles.

3. to be able to identify the corresponding parts of similar triangles.

4. to be able to solve indirect measurement problems involving similar triangles.

5. to be able to use trigonometric ratios to solve indirect measurement problems involving right triangles.

6. to be able to use the Pythagorean Theorem to find the missing side of a right triangle.

Three types of problem sets, with answers, are included in this module. Progress Tests appear at the end of each section. These Progress Tests are always short with only four to six problems. The questions asked in Progress Tests always come directly from the material of the section immediately preceding the test.

Exercise Sets appear less frequently in the module. More problems appear in an Exercise Set than in a Progress Test. These problems arise from all sections of the module preceding the Exercise Set. The problems in Part I of the Exercise Sets are specifically chosen to match the objectives of the module. Part II contains challenge problems.

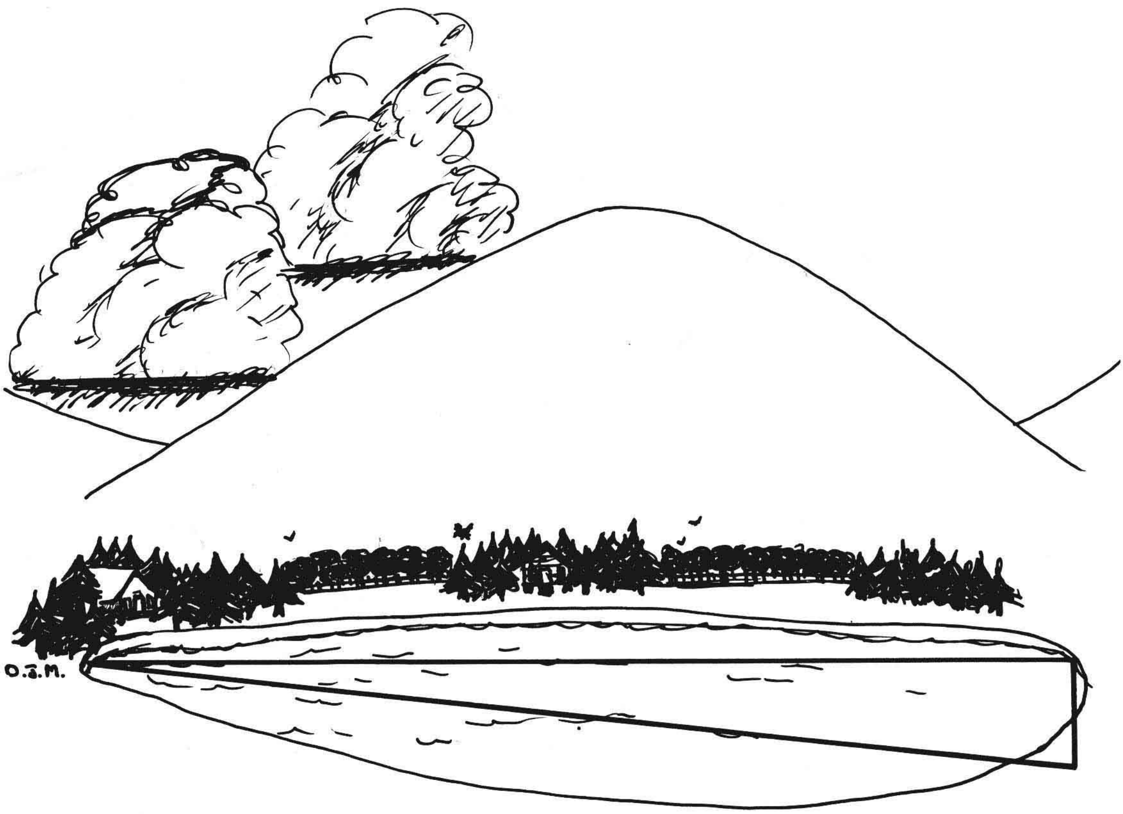
A Self-Test is found at the end of the module. Self-Tests contain problems representative of the entire module.

In learning the material, the student is encouraged to try each problem set as it is encountered, check all answers, and re-study those sections where difficulties are discovered. This procedure is guaranteed to be both efficient and effective.

Metric units of measure are used exclusively in this module. The necessary conversion facts to remember are:

1 kilometer = 1,000 meters

1 meter = 100 centimeters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INDIRECT MEASUREMENT

INTRODUCTION

To explain "indirect measurements" it is perhaps best to first describe what is meant by "direct measurements." The idea of direct measurements is the application of a measuring device directly to the object being measured. Consequently, a direct measurement of the length of a line segment might be obtained by placing a ruler or meter stick next to the line segment. The direct measurement of the weight of a piece of sirloin steak might be obtained by placing the particular piece of meat on a scale. The direct measurement of an angle's size in degrees might be obtained by applying a protractor to the angle itself.

Indirect measurements are involved when the measuring device is not applied directly to the object being measured. Measuring the height of a mountain, the distance to the moon, the amount of blood in a human body, or the weight of a building cannot be accomplished by direct measurement. Other measuring methods must be devised to make such measurements and these other methods are called indirect measurements.

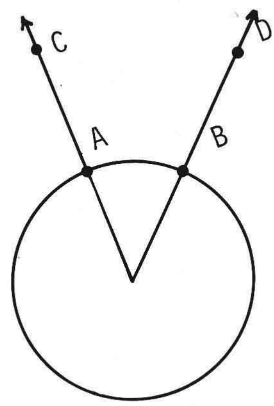
In this module, the indirect measurements will be limited primarily to measures of length and sizes of angles. However, two interesting historical examples of indirect measurements will be presented in this section to more completely explain the meaning of indirect measurements. Both of these historical examples are from the third century BC and involve famous Greek mathematicians.

The first story involves the great Greek mathematician Archimedes. Supposedly, the King had ordered an intricate gold crown. When the crown was delivered the King was suspicious that the crown was not pure gold. The King's problem was to determine whether the crown was pure gold without destroying or marring it. The King asked Archimedes for a solution.

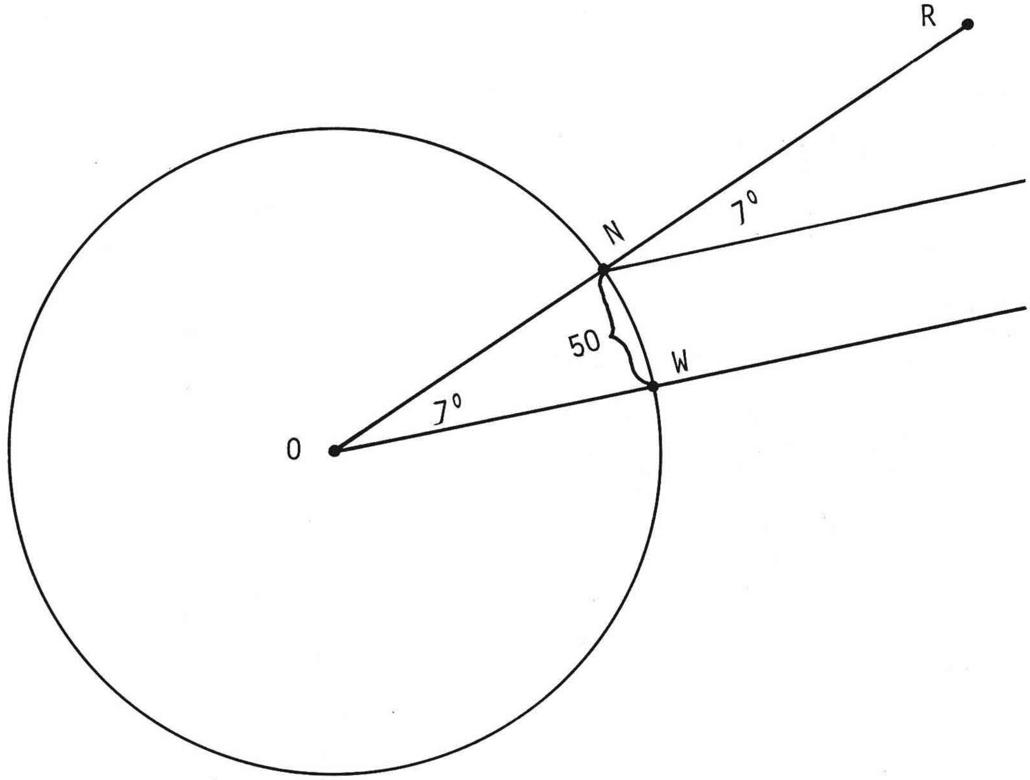
Archimedes pondered the problem without finding an immediate solution. Later, while taking a bath he suddenly realized that the water level in the tub changed according to the portion of his body that was immersed. The story goes that Archimedes jumped from the tub and ran through the streets shouting "Eureka, I've got it!" By measuring the amount of water displaced by the crown he could find the crown's volume and compare the weight of that volume of gold with the weight of the crown. He had found a way to "indirectly" measure the amount of gold in the crown.

The second historical example of an indirect measure involves the Greek mathematician Eratosthenes. Eratosthenes is credited with measuring the circumference of the earth 1700 years before Columbus convinced many Europeans that the earth was round.

Eratosthenes accurately measured the angle of the sun's rays to two different points of the earth. To understand Eratosthenes' method it is necessary to note that "straight up" is two different directions at two different positions on the earth's surface. "Straight up" at point A in the figure is the ray from A through C. "Straight up" at point B in the figure is the ray from B through D. The two rays are not in the same direction.



Eratosthenes determined the time when the sun was directly above one point of the surface of the earth by looking into a deep water well and noting the time when the sun shone directly into the water. He went due north a distance of 50 stadia, a unit of Greek linear measurement, and measured the angle between the sun's rays and the "straight up" line at that point. This angle measured approximately 7 degrees. In the figure on page 3, the point W represents the location of the well. The point N represents the spot on the earth's surface 50 stadia from the well. The line through N and R is the "straight up" line at N. The parallel lines through W and N represent rays of the sun when the sun is directly above the well.



Eratosthenes used the 7 degree angle measured at point N and the fact that the sun's rays at points N and W were parallel to show that the angle at the center of the circle, 0, was also 7 degrees. Geometry students may remember that corresponding angles are equal when parallel lines are crossed by a transversal. The 50 stadia arc from W to N therefore represented $\frac{7}{360}$ of the circumference of the earth because a full circle has 360 degrees.

Written as a proportion, the known and unknown numbers from Eratosthenes calculations would be:

$$\frac{7}{360} = \frac{50}{C} \text{ or } \frac{360}{7} = \frac{C}{50}$$

Eratosthenes solved the proportion and found the circumference of the earth in stadia. It is believed that his answer, if converted to kilometers, would be approximately 40,220 kilometers, which is quite accurate.

Progress Test 1

1. The volume of a statue was found by immersing the object in a rectangular tub of water and noting that the water level rose 10 centimeters. If the tub was 1.5 meters long and 0.9 meters wide, what was the volume of the statue in cubic meters? (The formula for the volume of a rectangular solid is $V = \ell wh$ where ℓ stands for the length, w for the width, and h for the height.)

2. If the statue weighed 1,200 kilograms, use the volume found in problem 1 to find the weight per cubic meter.

3. Two cities are located on the same longitudinal (north-south) line. Their latitudes differ by 9 degrees and the cities are 900 kilometers apart. Draw a figure showing these measurements.

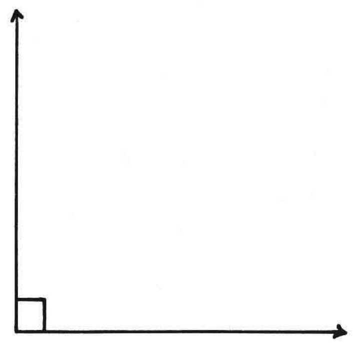
4. One proportion that could be written for the situation in problem 3 is $\frac{9}{360} = \frac{900}{C}$ where C represents the circumference of the earth. Write the other proportion using the fraction $\frac{360}{9}$.

SOME NECESSARY NAMES, SYMBOLS, AND NOTATIONS

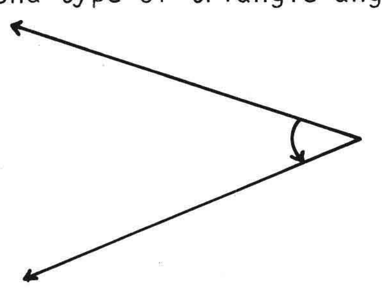
Before the indirect measuring concepts of this module can be completely represented, some names, symbols, and notation devices need to be understood.

The most important geometric figure involved in indirect measurement is the triangle. Every triangle contains three sides and has three angles, so sides of triangles and triangle angles also are important.

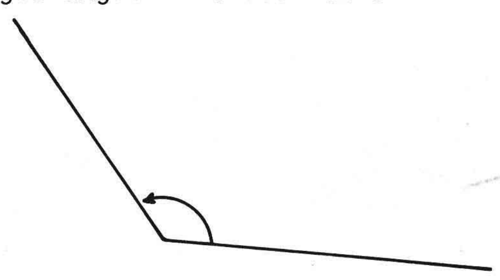
Three types of triangle angles are named by their relative sizes. A right angle is an angle in which the two sides of the angle (rays) are perpendicular to each other. The measurement of a right angle, in degrees, is 90^0. A right angle is shown in the figure. A small box is often used in the figure of a right angle to indicate that the sides are perpendicular.



A second type of triangle angle is called an acute angle. An acute angle is one with a measurement less than 90^0. An acute angle is smaller than a right angle in terms of its measurement, but this has nothing to do with the length of its sides (rays). An acute angle is shown in the figure accompanying this paragraph.



The third and last type of triangle angle is called an obtuse angle. An obtuse angle has a measure greater than 90^0. It is greater than a right angle in terms of its measurement. The figure accompanying this paragraph shows an obtuse angle.



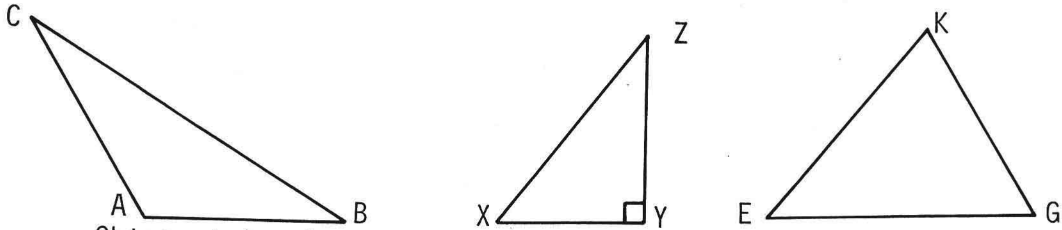
The sum of the degree measurements of the three angles of a triangle is always 180^0. This fact means that a triangle may have:

1. One obtuse angle and two acute angles.

2. One right angle and two acute angles.

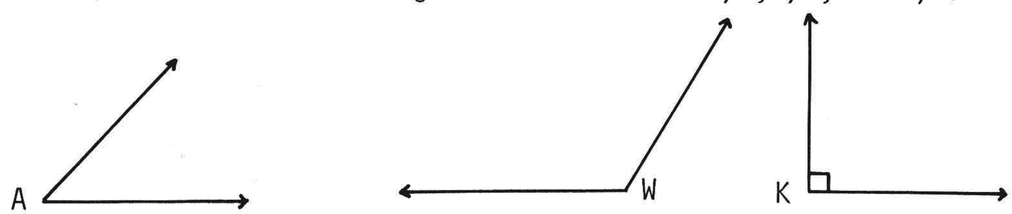
3. Three acute angles.

One way of naming triangles is by the size of their largest angle. Consequently, a triangle with an obtuse angle is called an obtuse triangle, a triangle with a right angle is called a right triangle, and a triangle with three acute angles is called an acute triangle. The obtuse triangle below is symbolized as $\triangle ABC$. The right triangle is named $\triangle XYZ$. The acute triangle is shown by $\triangle EGK$.

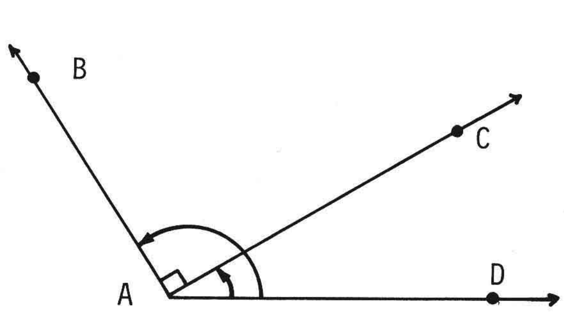


Obtuse triangle   Right triangle   Acute triangle

Each angle consists of two rays, called the sides of the angle, with a common endpoint which is called the vertex of the angle. There are two different uses of symbols for naming angles in this module. The symbol that is preferred in this module names an angle by the point which is its vertex. In the figure below the angles from left to right would be named $\measuredangle A$, $\measuredangle W$, and $\measuredangle K$.



Sometimes a point serves as the vertex of two or more angles. In such cases the naming of the angle by its vertex would be confusing. In the figure below, there are three angles with the vertex A. One of the angles is acute, another is a right angle, and the third is an obtuse angle.



Since point A in the figure is the vertex of three angles it would be confusing to name any of them ∢A. In cases where a point is the vertex of more than one angle, the angles are named by three points, one from each ray and the other from the vertex. The three points are listed with the vertex as the second point in the list. The symbols of the acute angle above would be ∢CAD or ∢DAC. The symbols for the right angle are ∢CAB or ∢BAC. The symbols for the obtuse angle are ∢BAD or ∢DAB.

Progress Test 2

Use the figure shown at the right to answer the questions.

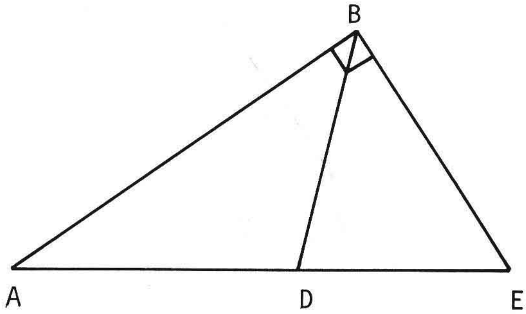
1. The right triangle is named ____________ .

2. ∢E is a(n) ________ angle.

3. The sum of the measurements of ∢A, ∢ADB, and ________ is 180^0.

4. Do the symbols ∢E, ∢BED, and ∢AEB name the same angle?

5. Which of the following symbols should not be used as the name of an angle: ∢DAB, ∢EAB, ∢D, ∢A?

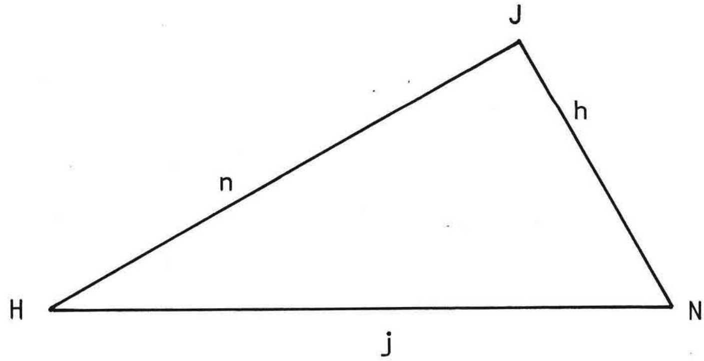


NAMING PARTS OF TRIANGLES

In the preceding section symbols for the angles of a triangle were explained. In this section, symbols for the sides and the measurements of the sides of a triangle are explained. Also discussed in this section are some of the relationships between angles and sides of triangles and the names given those relationships.

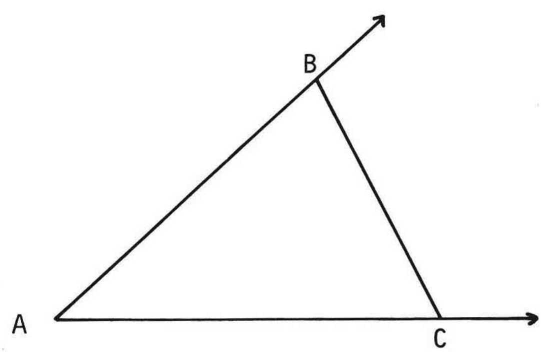
The sides of a triangle are line segments. To name the line segment joining the points A and B, the symbols $\overline{AB}$ or $\overline{BA}$ are used.

For the triangle shown below, the three sides may be symbolized by $\overline{HJ}$, $\overline{JN}$, $\overline{NH}$.

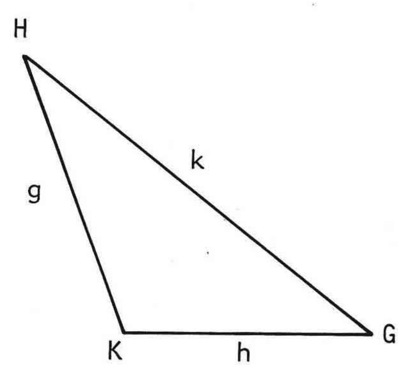


The small letters h, j, and n are shown beside the line segments $\overline{JN}$, $\overline{NH}$, and $\overline{HJ}$. Small letters are used to indicate the length of the line segments. Consequently, the measurement of $\overline{JN}$ is indicated in the figure as h units long. Capital letters such as A, H, and X are used throughout this module to indicate points. Lower case letters such as h, a, and x are used throughout this module to indicate linear measurements.

Each angle in a triangle has two sides of the triangle which lie on the rays of the angle and share the vertex of the angle. In the figure to the right, the rays of $\measuredangle A$ contain the sides $\overline{AB}$ and $\overline{AC}$ and share the vertex point A with those sides. The sides $\overline{AB}$ and $\overline{AC}$ are called adjacent sides for $\measuredangle A$. The side $\overline{BC}$ is called the opposite side to $\measuredangle A$. Each angle in a triangle has two adjacent sides and one opposite side.



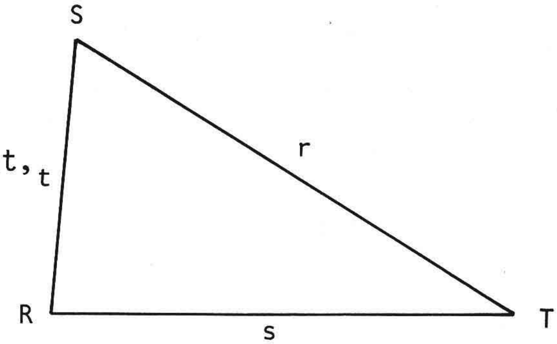
If the comparative sizes of the three angles of a triangle are known, then the comparative sizes of the three sides of the triangle can be found. In the figure to the right, $\measuredangle K$ is greater than $\measuredangle G$ which is greater than $\measuredangle H$. The sides opposite these three angles will have a similar size relationship. That is, k is greater than g which is greater than h. The symbol ">" is read



"greater than" and the size relationships of the triangle may be written as:

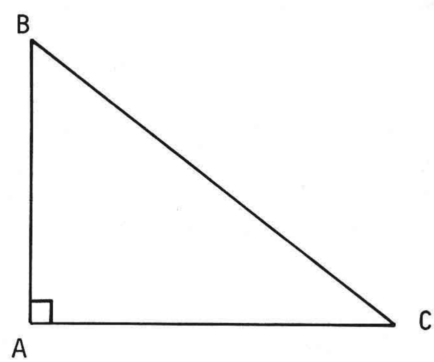
$$\sphericalangle K > \sphericalangle G > \sphericalangle H \text{ and } k > g > h$$

The angle-side relationship of the preceding paragraph is reversible. If the relative sizes of three sides of a triangle are known, then the relative sizes of the angles opposite those sides can be found. For the triangle shown at the right, if $r > s$ and $s > t$ then $\sphericalangle R > \sphericalangle S$ and $\sphericalangle S > \sphericalangle T$.



The angle-side size relationships have two special situations in those cases where triangles have two or more equal angles or where the triangle is a right triangle. When a triangle has two equal angles the sides opposite those angles must also be equal. Conversely, if two sides of a triangle are equal length, then the opposite angles must have the same measure. Extending this idea to a triangle with three equal angles leads to the conclusion that it also has three equal sides. Triangles with two equal sides and two equal angles are called isosceles triangles. Triangles with three equal angles and three equal sides are called equiangular or equilateral triangles.

In a right triangle the side opposite the right angle is always the longest side of the triangle because the right angle is the largest angle of the triangle. This longest side of a right triangle is given a special name and is called the hypotenuse. In the figure to the right, the hypotenuse is side $\overline{BC}$. Notice that $\overline{BC}$ is also an adjacent side for the acute angles, $\sphericalangle B$ and $\sphericalangle C$. Whenever a right triangle is being considered, the longest side is called the hypotenuse rather than an adjacent side to one of the acute angles. The term, adjacent side, is reserved for one of the shorter sides of a right triangle. Consequently, $\sphericalangle B$ in the drawing has $\overline{AB}$ as its adjacent side, $\overline{AC}$ is its opposide side, and $\overline{BC}$ is its hypotenuse. In a right triangle, and only in a right triangle, an acute angle has only one adjacent side.



Progress Test 3

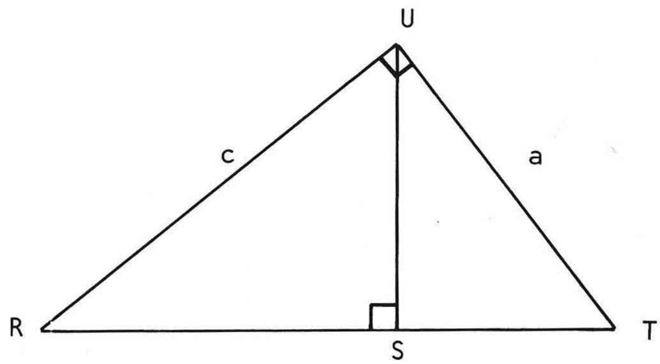
Use the figure shown to answer each question.

1. In $\triangle RTU$ what side is adjacent to $\angle R$?

2. In $\triangle SUT$ what side is opposite to $\angle T$?

3. If $c > a$ what is the size relationship of $\angle R$ and $\angle T$?

4. In $\triangle RUS$, with respect to $\angle RUS$, $\overline{UR}$ is called __________ , $\overline{US}$ is called __________ , and $\overline{RS}$ is called __________ .



Exercise Set 1

I. 1. $\triangle ABC$ has the following angles: $\angle A = 100^0$, $\angle B = 50^0$.

   a. $\triangle ABC$ is called a(n) __________ triangle.

   b. $\angle C$ measures __________ .

   c. The longest side is __________ .

2. $\triangle DEF$ has the following angles: $\angle D = 37^0$, $\angle E = 90^0$.

   a. $\triangle DEF$ is called a(n) __________ triangle.

   b. $\angle F$ measures __________ .

   c. The shortest side is __________ .