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TEXTS IN APPLIED MATHEMATICS

Mathematical Systems Theory I

Modelling, State Space Analysis, Stability and Robustness

Springer

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Modelling, State Space Analysis, Stability and Robustness

With 180 Figures

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Preface

The origins of this book go back more than twenty years when, funded by small grants from the European Union, the control theory groups from the universities of Bremen and Warwick set out to develop a course in finite dimensional systems theory suitable for students with a mathematical background, who had taken courses in Analysis, Linear Algebra and Differential Equations. Various versions of the course were given to undergraduates at Bremen and Warwick and a set of lecture notes was produced entitled "Introduction to Mathematical Systems Theory". As well as ourselves, the main contributors to these notes were Peter Crouch and Dietmar Salamon. Some years later we decided to expand the lecture notes into a textbook on mathematical systems theory. When we made this decision we were not very realistic about how long it would take us to complete the project. Mathematical control theory is a rather young discipline and its foundations are not as settled as those of more mature mathematical fields. Its basic principles and what is considered to be its core are still changing under the influence of new problems, new approaches and new currents of research. This complicated our decisions about the basic outline and the orientation of the book. During the period of our writing, problems of uncertainty and robustness, which had been forgotten for some time in 'modern control', gradually re-emerged and came to the foreground of control theory. Convinced of their key importance we finally deemed it necessary to make them a central subject of the book. Indeed we had already worked on problems of uncertainty ourselves, trying to develop tools for their analysis in state space theory where they had been largely neglected in the aftermath of geometric control theory. Our endeavour to develop a mathematical framework for dealing with such problems, both in the analysis and in the synthesis of control systems, brought up new research problems, and this interaction between the work on the book and work on research further delayed its completion.

Our aim has been to give a rigorous and detailed mathematical treatment of the basic elements of systems theory which could serve as a reference. But we also wanted to do justice to the origins of the subject in engineering and illustrate its interdisciplinary character by many examples and discussions on aspects of application. With this in mind we decided at an early stage that the book should be focussed on finite dimensional time-invariant linear systems. There were two main reasons for this choice. Firstly, nearly all the main problems, concepts and approaches in the theories of nonlinear and infinite dimensional control have their origins in linear finite dimensional theory. Secondly, advanced theories require more sophisticated mathematics, and there is the risk that technical problems of mathematics obscure the system theoretic content. This was in conflict with our wish to write a book accessible to students of mathematics after two years of study and to concentrate on the main issues and fundamental concepts of systems theory. Nevertheless, in spite of the focus on finite dimensional linear systems we have made it a rule to develop the basic system theoretic notions in full generality. Throughout the book the presentation proceeds in a systematic way from the abstract to the concrete. The exposition is restricted to time-invariant linear systems only where a development for other classes of systems would require advanced mathematical tools beyond those outlined in the appendix. For instance, we do not touch on any topics of nonlinear systems and control theory which require the use of differential geometric tools, nor do we deal with infinite dimensional systems theory since then a substantial preparation in functional analysis would be necessary.

The first two chapters of this volume are of an introductory nature whereas the others are more demanding and prepare the reader for research. The rigorous mathematical treatment is complemented by many examples, illustrations and explanatory comments. Also computational issues are discussed. As such, we hope the volume will be useful for established researchers in systems theory as well as those just starting in the field. For teaching it can be used at two different levels. The material can be filtered to obtain undergraduate courses, and individual graduate courses can be based on single or pairs of chapters. Indeed we have based undergraduate courses on Chapter 3, graduate ones on Chapters 3, 4, and Chapters 4, 5 and a seminar on Chapter 1. It is our experience that a first course in mathematical systems theory in the third year of a mathematics curriculum is an excellent way of showing students the usefulness of what they have studied in their first two years. In control theory they can learn that methods from different mathematical fields, like analysis, linear algebra, differential equations, complex analysis, integral transformations and numerical analysis, which they have studied separately in their first years, must be combined to develop a successful theory for applications.

The book is divided into two volumes. The second one will be concerned with *control* aspects and contains chapters on controllability and observability, input-output systems, geometric control theory, the linear quadratic problem and H_∞ control theory. The present first volume consists of five chapters and is concerned mainly with *systems analysis*. At the end of this volume there is a detailed index preceded by a glossary and an extensive bibliography. Every chapter, with the exception of the first, has the same format. Each is divided into sections and subsections with exercises and notes and references at the end of each section. Sections are numbered consecutively within chapters and subsections are numbered consecutively within sections. For example, Section 5.3 is the third section in Chapter 5 and Subsection 5.3.1 is the first subsection in Section 5.3. Theorems, propositions, definitions etc. are numbered consecutively by chapter and section in a single list and are indexed with three numbers. Thus Theorem 5.1.8 refers to a theorem in Section 1 of Chapter 5 and is the eighth theorem or example etc. in the list of that section. Figures and tables are numbered consecutively, e.g. Figure 4.1.7 could be followed by Table 4.1.8. Equations are numbered by single numbers in each section, and are referenced by this number in the section where it occurs. For example (9) refers to the ninth equation in the same section. However, within say Chapter 3, the ninth equation in Section 2 is written (2.9) when cross-referenced in say Section 3,

whereas, if the equation is referred to in any other chapter we give the triple (3.2.9). Exercises are referenced in a similar way, i.e. we write Ex. 9, Ex. 2.9 or Ex. 3.2.9.
A survey of the material in each of the chapters can be obtained by looking at the table of contents. Below we give a brief overview.
The first chapter is of an illustrative and motivational character. It presents a series of dynamic models from six areas of application and explains by examples how dynamic phenomena in different fields of science and engineering can be translated into appropriate mathematical representations. It also shows how typical system theoretic problems and concepts arise in these fields. The descriptive style adopted in this chapter is rather different from the mathematical style of the ensuing chapters. Most of the sections just give a catalogue of examples from the corresponding field of application. The sections on *mechanics* and *electromagnetism* are different. These fields have their own well-established theories of dynamics. In fact control theory has emerged from mechanical and electrical engineering which are still the main areas of application. We therefore deemed it appropriate to explain some of the scientific principles behind the dynamic models in these areas and sketch some modelling techniques in use. Altogether, the chapter is meant as an introduction to dynamic models and an illustration of the diversity of dynamical phenomena to which system theoretic concepts may be applied. Some of the models described here are taken up later in the examples of the following chapters.
The introduction to mathematical systems theory begins with Chapter 2. Some readers may prefer to start directly with this chapter and go back to Chapter 1 for more details whenever an example from the first chapter is used for illustration. Chapter 2 provides an introduction to state space theory. We have chosen to use the input-state-output approach put forward by Kalman. The general concept of a dynamical system is developed and then it is specialized to the linear case. Continuous time and discrete time systems are treated in parallel and are interrelated by a discussion of sampling and approximations problems. Some preliminary elements of input-output theory are also introduced and the relationship between the analysis of input-output systems in time and in frequency domain is explained.
The next chapter deals with stability theory. Some elements of topological dynamics and Liapunov's stability theory are developed in a general setting and then specialized to different classes of systems. A notable feature of this chapter is that the sections on Liapunov's analytical approach are complemented by an extensive final section on classical algebraic stability theory.
One would expect to find some of the material of the previous chapters in a book on systems theory, but the inclusion of a chapter on perturbation theory (the subject of Chapter 4) might seem surprising. We felt it was necessary because many of the results we give permeate various branches of systems theory but are rarely explicitly stated and proved in books on systems and control. Moreover we wished to address the robustness question in a general setting and so needed to introduce some elements of μ-analysis.
The final chapter of this first volume reflects our joint research on uncertain systems. Our main objective is to develop a spectral theory for uncertain time-invariant linear systems. We do this via *spectral value sets* and *stability radii* and most of the chapter is devoted to deriving both qualitative and quantitative results for them.

However we also deal with the problem of transient deviations of trajectories from an equilibrium point and in a final section obtain results for stability radii of uncertain systems with respect to time-varying, nonlinear and dynamic perturbations.
Since the range of mathematics used in this volume is quite wide we have included some of the background mathematics in fairly substantial appendices.
We have tried our best to eliminate any errors in the book. However our experience has shown that this is a never ending process and we would be very grateful if readers could communicate to us any errors and inaccuracies they encounter in this volume.

In conclusion we would like to thank those colleagues who helped us, directly or indirectly, with the preparation of this book. As students of mathematics we did not come into contact with systems theory. We learnt it whilst lecturing at university and have been strongly influenced by friends and colleagues who at an early stage in our careers introduced us to their fields of research during periods when they were guest professors of our universities or when we were invited to their research centres. We benefited greatly from their knowledge and advice, and would like to express our special thanks to Roger Brockett, Chris Byrnes, Ruth Curtain, Paul Fuhrmann, Michiel Hazewinkel, Michael Heymann, Alan Laub, Larry Markus, Howard Rosenbrock, Jan Willems, Murray Wonham and Jerzy Zabczyk. We also owe thanks to our doctoral students and co-workers at that time, who are now friends and colleagues. Their enthusiasm and manifold contributions spurred our research and without them we would not have undertaken this project.
More recently, we have profited from the expertise of the many people who visited us in Bremen and Warwick. In particular we are indebted to Vladimir Kharitonov. His series of lectures on algebraic stability theory in Bremen helped us with the preparation of Section 3.4. Our doctoral students and colleagues Eduardo Gallestey, Michael Karow, Elmar Plischke and Fabian Wirth have collaborated with us in the research which led to the results presented in Chapter 5. Many of the examples and figures in this chapter are due to them. Fabian read some of the sections and made suggestions for their improvement. We also would like to thank Buddug Pritchard who helped us with the English. In the early days Bernd Kelb typed some of the sections, computed some of the examples, constructed some of the figures and helped us with LaTeX. More recently Elmar has taken on this role. Not only has he contributed in research to the development of the material on transient behaviour in Chapter 5, he has also computed many figures and read, and suggested improvements to many of the sections. Moreover he has been a rock for us with his technical knowledge of and expertise with the computer. Whenever we had problems with Unix, Linux, LaTeX, xfig, MATLAB he willingly gave us his assistance and always did so with a wry sense of humour. Finally we would like to thank the team at Springer, in particular Ruth Allewelt and Martin Peters who have been most helpful, patient and understanding.

Bremen
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Contents

Chapter 1

Mathematical Models

In this chapter we present a range of dynamical systems from different areas of application and use them as examples to illustrate some typical problems from systems and control theory. Several of the mathematical models we introduce and discuss in the following sections will be taken up as examples in later chapters.
The development of mathematical systems theory starts in the next chapter. Readers who prefer to go directly to Chapter 2 can do so without any difficulty as the mathematical exposition in that chapter is self-contained and independent of following material. On encountering an example based on a dynamic model from Chapter 1, they may wish to look back to its origin here to find more details and get additional background information.
This chapter consists of six sections in which we present dynamical models from the following areas:

- Biology (Population Dynamics)
- Economics
- Mechanics
- Electromagnetism and Electrical Systems
- Digital Systems
- Heat Transfer

The mathematical models in the first three sections are described by *ordinary differential equations* and by *difference equations*. Also in Section 1.4, although the basic equations of electromagnetism are *partial differential equations*, we will only consider so-called *lumped models* of electromagnetic devices which again are described by ordinary differential equations. Different types of models are presented in the remaining two sections. In Section 1.5 we consider digital systems which have only a finite number of different states and are represented as finite automata. In the last section we deal with an example of a distributed parameter system described by partial differential equations.
In all these sections we will not only discuss the mathematical models but also point out some of the problems encountered in determining a mathematical model for a real process. While most of the sections just present a gallery of typical examples, some modelling methods will be sketched out in the sections on mechanical and electrical systems.

1.1 Population Dynamics

In order to predict or estimate the growth of a given population one needs a dynamical model. Such models may also be useful if one wants to control the development of a population. For example problems of control arise in fisheries management where one would like to keep fishing at a sustainable level and maximize the average catch over long time periods. In other applications interaction between different populations may be important and one may make use it for control purposes, e.g. in pest control where one introduces predators to reduce the pest. In this section we consider two classical models of population dynamics.

Example 1.1.1. (Logistic growth model). The simplest growth model is

$$\dot{x}(t) = ax(t). \tag{1}$$

Here $x(t)$ is the size, density or biomass of a given population at time t and the growth parameter a is the *intrinsic growth rate* (difference between the birth rate and the death rate) of the population. If the initial size of the population is $x(0) = x_0 > 0$ the development follows the exponential law $x(t) = e^{at}x_0$. Thus we have exponential growth if $a > 0$ (i.e. the birth rate is larger than the death rate) and exponential decay if $a < 0$. The idea that human populations when "unchecked by the difficulties of subsistence" have a positive constant natural growth rate goes back to Malthus. In his *Essay on Population (1798)* he contrasted the natural geometric growth of mankind with the linear growth of subsistence resources and drew far reaching conclusions from this which had a profound effect on political economics.

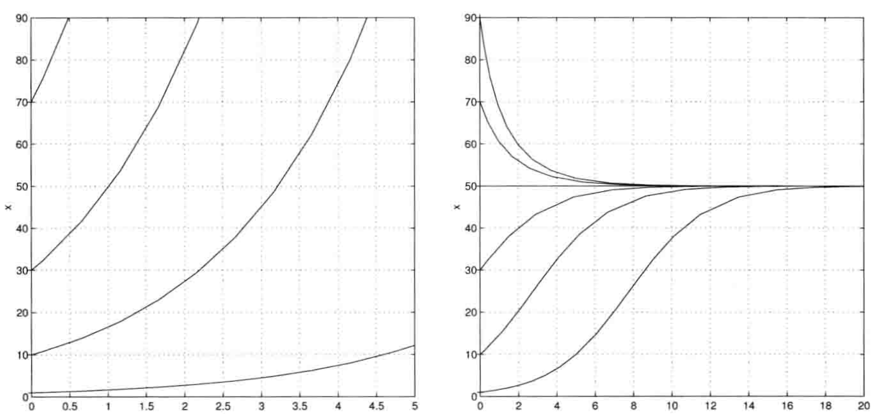

Figure 1.1.1: Exponential and logistic growth models

The exponential growth model, although adequate in many applications over a limited time span becomes unrealistic in the long run since $e^{at}x_0 \to \infty$ as $t \to \infty$. The growth rate $\dot{x}(t)/x(t)$ cannot be constant over arbitrarily long periods of time, since resources are limited. As the population becomes larger and larger, restraining factors will have an increasingly negative effect on population growth ("crowding"). In 1838 Verhulst proposed another growth model which incorporated the limiting factors and accounted for the fact that individuals compete for food, habitat, and other limited resources,

$$\dot{x}(t) = r(K - x(t))x(t). \tag{2}$$

According to this model a small population will initially grow at an exponential rate rK but as the population increases the growth rate will be diminished.
If the system is initially at $x_0 = K$ then it will remain at $x(t) = K$ for all time. Then the population is at an equilibrium $x(t) \equiv \bar{x} = K$, $t \geq 0$. If $0 < x_0 < K$ the population $x(t)$ will increase continuously and approximate K as $t \to \infty$. If $x_0 > K$, the population size $x(t)$ will converge towards K from above. In fact the following formula for the solution is easily obtained by separation of variables

$$x(t) = \frac{K}{1 + (K/x_0 - 1)\, e^{-rKt}}.$$

The graphs of these solutions are called *logistic curves* and Verhulst's model is also known as the *logistic growth model.* Figure 1.1.1 illustrates that $x(t) \equiv K$ is a stable equilibrium, i.e. all trajectories with initial state $x_0 > 0$ converge towards this equilibrium as $t \to \infty$. The saturation level K is interpreted as the *environmental carrying capacity* of the corresponding ecosystem.Now suppose that we want to describe the dynamics of a fish population under the influence of fishing. If $u(t) \geq 0$ is the catch rate and we assume the logistic growth model for the undisturbed fish population, we obtain *Schaefer's model*

$$\dot{x}(t) = r(K - x(t))x(t) - u(t). \tag{3}$$

Note that only non-negative solutions $x(t, u) \geq 0$ make sense. Given an initial state $x_0 > 0$ and a fixed time period $[t_0, t_1]$, a fishing policy $u(\cdot) : [t_0, t_1] \to \mathbb{R}_+$ may be called "admissible" if it leads to a non-negative solution $x(t, u)$ of (3) for $t \in [t_0, t_1]$ and "optimal" if it maximizes the overall catch during that period. Such an "optimal" fishing policy will, however, lead to depletion at time t_1. To prevent this one may wish to impose a "terminal constraint" $x(t_1) \geq x_1$ where $x_1 > 0$ is a lower bound to an acceptable fish population at the end of the period. Thus we end up with the following *optimal control problem:*

$$\text{Maximize} \int_{t_0}^{t_1} u(t)dt \;\text{ subject to }\; u(t) \geq 0,\, x(t, u) \geq 0,\, t \in [t_0, t_1],\, x(t_1) \geq x_1.$$

If $u(t)$ is required to be constant, the problem is easily solved, see Ex. 2.1.15.
Another optimal control problem which can be solved by elementary means is the *optimal constant-effort harvesting problem.* Here the harvesting rate $u(t)$ is by definition proportional to $x(t)$, i.e. $u(t) = cx(t)$. This is a simple example of *feedback control* where the control variable $u(t)$ is determined as a given function of the instantaneous state $x(t)$ of the system. Following this control strategy one obtains a Verhulst model in which the parameters have changed

$$\dot{x}(t) = r(K - c/r - x(t))x(t).$$

If $c < rK$ there is an equilibrium solution $x(t) = \bar{x} = K - c/r$, $t \geq 0$ corresponding to the constant harvesting policy $u(t) = c\bar{x}$, $t \geq 0$. Again one can determine the optimal constant harvesting policy which yields the highest sustainable harvesting rate, see Ex. 2.1.15. □

Remark 1.1.2. Although the logistic model is a widely used and successful model which predicts quite well the growth of various laboratory populations (see *Notes and References*), it is a highly simplified model. It is based on a number of assumptions which are not usually satisfied when the growth of a species in a real ecosystem is considered, e.g.

(i) The influence of environmental factors on the growth of the species is assumed to be constant in time. But these factors and the behaviour of a species usually vary with the time of the year. Also there are often random variations in the environment.

(ii) The effects of limited resources are assumed to affect all individuals of the species in an equal manner. A more realistic model would take the spatial distribution of the species and its resources into account (partial differential equations).

(iii) It is assumed that the birth and death rates of the population respond instantly to the population size, whereas usually there is a delay between birth and the ability to give birth.

(iv) The age distribution of the population is assumed to be constant or that if it changes it does not influence the growth of the species.

Although the assumptions are not realistic, highly simplified models like that of Verhulst are often of great scientific value. Their purpose is not to give an accurate portrait of an underlying real process but to enhance the understanding of some of its internal mechanisms. As such they can be more important motors for scientific progress than complex "realistic" simulation models[1]. □

Often the dynamics of a population are strongly influenced by the interaction with other populations in the same ecosystem. Several species may compete for the same natural resources or a species may be predatory on some species while serving as prey for others. In the following example we describe a classical predator-prey model due to Lotka and Volterra[2].

Example 1.1.3. (Predator-prey system). Suppose that an island is populated by goats and wolves. The goats survive by eating the island's vegetation and the wolves survive by eating the goats. Often oscillations are observed in the development of such predator-prey populations. If, initially, there are only a few wolves but many goats, the wolves have a lot to eat and the number of goats will be diminished while the number of wolves will increase until there are not enough goats to feed them. Then the number of wolves will be reduced so that the goats will be able to recover and this closes the cycle. The classical Lotka-Volterra model for such a predator-prey system is

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1 x_2 \\ \dot{x}_2 &= -cx_2 + dx_1 x_2, \end{aligned} \tag{4}$$

where x_1 and x_2 are the densities (number per unit area) of the prey and predator populations respectively, and a, b, c, d are positive constants. The model mirrors a qualitative feature which has been observed in many real predator-prey systems, the persistence of periodic fluctuations. This is illustrated in Figure 1.1.2. $\bar{x} = (c/d,\ a/b)$ is an equilibrium point of (4) and any initial state $x^0 \neq \bar{x}$, $x_1^0 > 0, x_2^0 > 0$ leads to a periodic trajectory cycling around this equilibrium point in the positive orthant.

Clearly, this is a simplistic model and does not aim at simulating or predicting a real process. The model is based on the following assumptions.

[1] "This work seeks to gain general ecological insights with the help of general mathematical models. That is to say the models aim not at realism in detail, but rather at providing mathematical metaphors for broad classes of phenomena. Such models can be useful in suggesting interesting experiments or data collecting enterprises, or just in sharpening discussion." (R. M. May, Preface of "Stability and Complexity in Model Ecosystems").

[2] The story of how Volterra came to design the model (independently of Lotka) is interesting. For many years fishermen had observed periodic fluctuations between sharks and their prey populations in the Adriatic Sea. During World War I, commercial fishing was greatly reduced and so it was expected that there would be plentiful fish stocks for harvesting after the war was over. Instead the catches of commercially valuable fish declined after the war while the number of sharks increased.

(i) In the absence of predators the prey population grows exponentially with rate a.

(ii) In the absence of prey the predator population decreases at the death rate c.

(iii) The growth of the predator population depends affinely on the food intake, i.e. on predation.

(iv) Predation depends on the likelihood that a victim is encountered by a predator and this likelihood is proportional to the product $x_1 x_2$ of the two populations' densities.

An assumption similar to (iv) is made in chemical kinetics where, according to the so-called law of mass action, the rate of molecular collisions of two substances in a given solution is assumed to be proportional to the product of their concentrations.

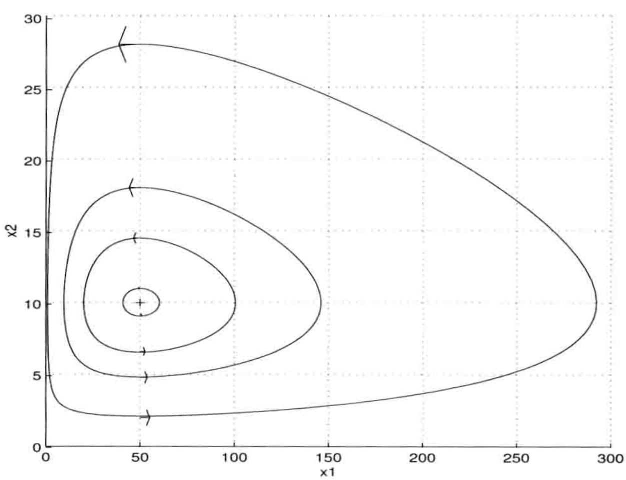

Figure 1.1.2: Predator-prey trajectories

Many "more realistic" models have been obtained from (4) by modifying the predator-free prey growth term ax_1 to include crowding effects or by allowing for saturation effects and lags in the predators' response to increasing prey densities. For instance, in order to eliminate the assumption that the prey grows exponentially in the absence of predators one could introduce a term $-ex_1^2$ in the first equation of (4) which accounts for the effect of crowding on the growth of the prey (see Example 1.1.1).

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1 x_2 - ex_1^2 = e(a/e - x_1)x_1 - bx_1 x_2 \\ \dot{x}_2 &= -cx_2 + dx_1 x_2. \end{aligned} \tag{5}$$

This drastically alters the qualitative behaviour of the predator-prey system. In the absence of predators the prey now evolves according to a logistic growth model with carrying capacity a/e. Moreover, the new system does not always have an equilibrium with positive coordinates. In fact the equilibrium equations are

$$(a - bx_2 - ex_1)x_1 = 0, \quad (-c + dx_1)x_2 = 0$$

and these equations have a (unique) positive solution $\bar{x} = (c/d, (da - ec)/bd)$ if and only if $a/e > c/d$. Figure 1.1.3 illustrates the changed behaviour of the modified predator-prey system (5). In particular, it has no non-constant periodic solutions and its only