Mathematics for Economics

second edition

Michael Hoy
John Livernois
Chris McKenna
Ray Rees
Thanasis Stengos

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This book is dedicated to our families:

Maureen, Alexis, and Joshua
Brenda, Ali, and Becky
Jane, James, and Kate
Denny, Zac, and Dan
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Preface

A major challenge in writing a book on mathematics for economists is to select the appropriate mathematical topics and present them with the necessary clarity. Another challenge is to motivate students of economics to study these topics by convincingly demonstrating their power to deal with economic problems. All this must be done without sacrificing anything in terms of the rigor and correctness of the mathematics itself.

A problem lies in the difference between the logic of the development of the mathematics and the way in which economics progresses from models of individual consumer and firm, through market models and general equilibrium, to macroeconomic models. The primary building blocks, the models of consumer and firm behavior, are based on methods of constrained optimization that, mathematically speaking, are already relatively advanced. In this book we have chosen instead to follow the logic of the mathematics. After a review of fundamentals, concerned primarily with sets, numbers, and functions, we pay careful attention to the development of the ideas of limits and continuity, moving then to the calculus of functions of one variable, linear algebra, multivariate calculus, and finally, dynamics. In the treatment of the mathematics our goal has always been to give the student an understanding of the mathematical concepts themselves, since we believe this understanding is required if he or she is to develop the ability and confidence to tackle problems in economic analysis. We have very consciously sought to avoid a "cookbook" approach.

We have tried to develop the student's problem-solving skills and motivation by working through a large number of examples and economic applications, far more than is usual in this type of book. Although the selection of these, and the order in which they are presented, was determined by the logic of the development of the mathematics rather than that of an economics course, in the end the student will have covered virtually all of the standard undergraduate mathematical economics syllabus.

Many people helped us in the preparation of this book and it is a pleasure to acknowledge our debt to them here. The following individuals read early versions of the manuscript and offered helpful suggestions, a large number of which were freely used:

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Part I Introduction and Fundamentals

Chapter 1 Introduction

Almost for as long as economics has existed as a subject of study, mathematics has played a part in both the exploration and the exposition of economic ideas.[1] It is not simply that many economic concepts are *quantifiable* (examples include prices, quantities of goods, volume of money) but also that mathematics enables us to explore relationships among these quantities. These relationships are explored in the context of *economic models,* and how such models are developed is one of the key themes of this book. Mathematics possesses the accuracy, the rigor, and the capacity to deal clearly with complex systems, which makes it highly valuable as a method for analyzing economic issues.

This book covers a wide range of mathematical techniques and outlines a large number of economic problems to which these techniques may be applied. However, mathematical modeling in economics has some unifying features and conventions that we will summarize here at the outset. Although model details are problem-specific, there are some basic principles in the modeling process that are worth spelling out.

1.1 What Is an Economic Model?

At its most general, a model of anything is a *representation.* As such, a model differs from the original in some way such as scale, amount of detail, or degree of complexity, while at the same time preserving what is important in the original in its broader or most salient aspects. The same is true of an economic model, though unlike model airplanes, our models do not take a physical form. Instead, we think of an economic model as a set of mathematical relationships between economic magnitudes. Knowing how to distill the important aspects of an economic problem into an abstract simplification is part of the formal training of an economist. The model must be convincing and must be capable of addressing the questions that the researcher has set. We now set out the central features of an economic model.

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[1] The more famous early works in economics with a mathematical exposition include A. Cournot, *Récherches sur les principes mathématiques de la théorie des richesses* (1838), W. S. Jevons, *The Theory of Political Economy* (1874), L. Walras, *Eléments d'économie politique pure* (1874), A. Marshall, *Principles of Economics* (1890), and V. Pareto, *Cours d'économie politique* (1896).

Quantities, Magnitudes, and Relationships

We can start by thinking of how we measure things in economics. Numbers represent quantities and ultimately it is this circumstance that makes it possible to use mathematics as an instrument for economic modeling. When we discuss market activity, for example, we are concerned with the quantity traded and the price at which the trade occurs. This is so whether the "quantity" is automobiles, bread, haircuts, shares, or treasury bills. These items possess *cardinality,* which means that we can place a definite number on the quantity we observe. Cardinality is absolute but is not always necessary for comparisons. *Ordinality* is also a property of numbers but refers only to the ordering of items. The difference between these two number concepts may be illustrated by the following two statements.

1. Last year, the economy's growth rate was 3%.
2. The economy's output last year was greater than the year before.

Both of these statements convey *quantitative* information. The first of these is a cardinal property of the change in output. We are able to measure the change and put a definite value on it. The second is an ordinal statement about economic activity in the past year. Last year's output is higher than the year before. This of course is an implication of the first statement, but the first statement cannot be inferred from the second statement.

However, there is a greater difference between cardinality and ordinality, because we can also decide on a *ranking* of items based on their *qualitative* properties rather than on their quantifiable ones. Most statements about preferences are ordinal in this sense, so to say: "I prefer brand A to brand B, and brand B to brand C" is an ordinal statement about how one person subjectively evaluates three brands of a good. If we let larger numbers denote more preferred brands, then we could associate brand A with the number 3, brand B with the number 2, and brand C with the number 1. However, the numbers 10, 8, and 0 would serve equally well in the absence of any other information. This statement may provide useful information, and certain logical and mathematical consequences may follow from it, but it is not a statement about quantities.

Variables and Parameters

When we start to build an economic model, we know that we are not going to be able to explain everything. Some things must be treated as given or as data for our problem. These are the *exogenous variables* and the *parameters* of the model. The *endogenous variables,* then, are those that are going to be explained by the model. A simple example will illustrate.

Suppose that we are trying to determine the equilibrium price and quantity in a market for a homogeneous good. We hypothesize that the quantity demanded of some good may be represented as

$$q^D = a - bp + cy \tag{1.1}$$

which is a simple linear demand function. Each time price, p, increases by one dollar, the quantity demanded, q^D, falls by b times one dollar. A rise in income, y, of one dollar increases quantity demanded by c units. This demand curve may be chosen purely for simplicity, or it may be known, by looking at market data, that the demand curve for this good does take this simple form. Now suppose that the supply of this good is fixed at some amount which we will call $\bar{q}^S$, and suppose that we believe that the prevailing price in this market is the price that equates demand with supply. Then

$$q^D = \bar{q}^S \quad \text{implies} \quad p = \frac{a - \bar{q}^S + cy}{b} \tag{1.2}$$

So here, p is the endogenous variable, a and b are the exogenously given parameters, and $\bar{q}^S$ and y are exogenous variables. For instance, demand is determined by tastes, weather, and many other environmental and social factors, all of which are captured here by a, b, and c. All supply-side considerations are, in this particularly simple case, summarized by the quantity $\bar{q}^S$. Parameters may also incorporate the effects of exogenous variables, which we do not wish to specify explicitly. For example, a may incorporate the effects of prices of others goods on the demand for this one. Finally, in this example, there is one further endogenous variable. Since quantity demanded, q^D, depends partly on p (which is endogenous) it too is endogenous: therefore q^D is only known when the price is known. Substituting equation (1.2) into equation (1.1) gives simply $q^D = \bar{q}^S$ as the value of demand.

In general, as in this simple example, we can use relationships between economic variables and background parameters to reach conclusions or predictions based upon the mathematical solutions of those relationships.

Behavior and Equilibrium

As we have just seen, a further step in building an economic model is to identify the *behavioral* equations, or the equations that describe the economic environment, and to identify the *equilibrium* conditions. In the simple supply-and-demand example above, the behavioral equations are the demand and supply functions describing the relationships between the endogenous variables and exogenous variables. The equilibrium condition determines what the values of the endogenous variables will be. In this case the condition is that supply equals demand. The specification of the equilibrium condition is based on our understanding of how the part of the economy in question works, and embodies the crucial hypothesis of how the endogenous variables are determined.

Behavioral equations contain hypotheses about the way the individual, market, or economy works. One of the key strengths of mathematical analysis in economics is that it forces us to be precise about our assumptions. If the implications of those behavioral equations prove to be unsubstantiated or ridiculous, then the natural course of action is to look more closely at the assumptions and to attempt to identify those responsible for throwing us off course.

Single-Equation Models and Multiple-Equation Models

Although sometimes the problem we are trying to analyze may be captured in a single-equation model, there are many instances where two or more equations are necessary. Interactions among a number of economic agents or among different sectors of the economy typically cannot be captured in a single equation, and a system of equations must be specified and solved simultaneously. We can extend our earlier example to illustrate this.

Consider first the demand and supply of two goods. We denote the demands by q_1^D and q_2^D, and the supplies by q_1^S and q_2^S, where the subscripts 1 and 2 identify which good we are referring to. Now, as before, we may specify how demands and supplies are related to the prices of the two goods, but this time recognizing that the demand for and supply of good 1 may depend on both its own price and on the price of good 2. Recognition of this fact gives rise to an interdependence between the two markets. For simplicity, suppose that the demand for each good depends on both prices, while the supply of each good depends only on the good's own price. The question we are asking is: If the interdependence between two goods takes this form, what are the consequences for the equilibrium price and quantity traded in each market in equilibrium? Again, as before, we will restrict ourselves to *linear* relationships only. We may write

$$q_1^D = a - b_1 p_1 + b_2 p_2, \qquad b_1, b_2 > 0 \tag{1.3}$$

and

$$q_2^D = \alpha - \beta_1 p_2 + \beta_2 p_1, \qquad \beta_1, \beta_2 > 0 \tag{1.4}$$

Notice that in addition to incorporating the usual negative relationship between the demand for a good and its own price, we have included a specific assumption about the *cross-price effects,* namely that these goods are *substitutes*. If the price of good 1 increases, the demand for good 2 increases, and vice versa. Setting supply equal to an exogenous amount in each market gives us