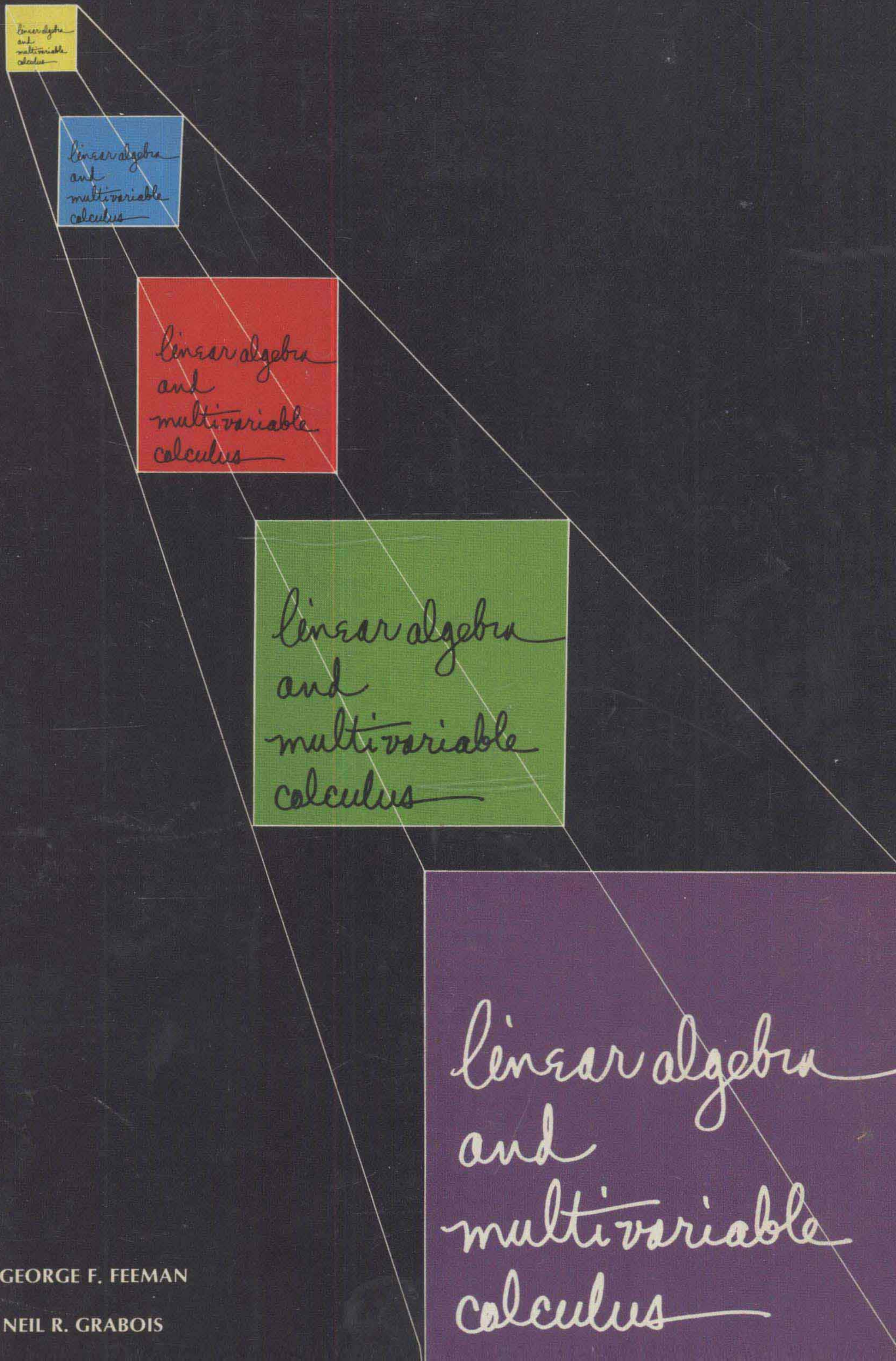

GEORGE F. FEEMAN

NEIL R. GRABOIS

GEORGE F. FEEMAN

Professor of Mathematics
Oakland University, Rochester, Michigan

NEIL R. GRABOIS

Associate Professor of Mathematics
Williams College, Williamstown, Massachusetts

McGRAW-HILL BOOK COMPANY

New York St. Louis San Francisco Dusseldorf
London Mexico Panama Sydney Toronto

This book was set in Optima
by Progressive Typographers,
and printed on permanent paper
and bound by The Maple Press Company.
The designer was Barbara Ellwood;
the drawings were done
by B. Handelman Associates, Inc.
The editors were Edward Millman
and Antonia Stires.
Les Kaplan supervised the production.

Linear Algebra
and
Multivariable Calculus

Library of Congress Catalog Card Number 69-18709

20337
1234567890 MAMM 79876543210

This book was written to provide an integrated one-year course in linear algebra and multivariable calculus and is intended to follow a standard course in one-variable calculus.

The topics have been organized so that the algebra and calculus are interwoven, reinforcing each other. Thus the role played by linear algebra in the calculus becomes clear, and abstract mathematical structures are introduced with proper motivation. Ideas in the multivariable calculus are given as natural restatements of their one-variable counterparts.

Although the topics were chosen with a view to the needs of students preparing for further work in mathematics, physics, and economics, they conform closely to the recommendations of the Committee on the Undergraduate Program in Mathematics (CUPM). This book forms an integrated version of Mathematics 3 and 4 as outlined in the pamphlet A General Curriculum in Mathematics for Colleges, prepared by CUPM in 1965.

The material has been arranged to provide motivation, review, and reinforcement of the central ideas. We have not tried to develop an important idea entirely in one section but have spread the development over several sections. For example, the notion of the jacobian matrix is essential to the study of nonlinear functions, since it allows them to be approximated by linear functions. This central idea is introduced in a computational way in Chap. 5, with reliance on the study of linear functions and matrices in Chap. 4; is used in Chap. 8 to study inverse and implicit functions; and appears again in a central role in the integral calculus of Chap. 9. Similarly, eigenvalues are introduced in the two-dimensional case in Chap. 6, are used in applications of the derivative in Chap. 8, and appear again in the study of invariant subspaces in Chap. 10.

Among the unusual features of the book are the use of column vectors for points in $\boldsymbol{R}^n$ to facilitate and simplify the development of matrix multiplication, the postponement of systems of linear equations to Chap. 7, the inclusion of a brief section on extreme values of integrals (really an introduction to the calculus of variations), and a section on the application of normal forms to systems of linear differential equations with an introduction to the exponential of a matrix.

Numerous examples have been included in the text. These examples, which are discussed in detail, illustrate either the application of an abstract result to a specific case or a technique for handling a general class of problems. Exercises have been interspersed throughout the text, frequently following examples. They form an integral part of the text and provide the student with an opportunity to test his comprehension as soon as a new idea has been introduced and illustrated. Each exercise should be solved in its entirety, at the time it appears in the text. Answers to all the exercises are included in the book.

Each section closes with an extensive set of problems, which are a mixture of computation and theory. Their purpose is to test the student's understanding and manipulative skills, as well as to introduce new ideas and applications. The problems range in difficulty from straightforward computations to challenging theoretical questions. Answers and hints to the odd-numbered problems are also included in the book. Problems marked with an asterisk are used or referred to in later sections.

Detailed proofs of all but a few deep theorems have been given. In the exceptional cases, such as the inverse-function theorem in Chap. 8, the theorems are discussed intuitively with examples, and references are supplied. The symbol ■■■ is used to denote the end of a proof.

The first six chapters should be covered in their entirety. After completion of Chap. 6, sections can be chosen to satisfy a variety of emphases. For example, one can proceed directly from Chap. 6 to Chap. 10 if more linear algebra is desired, or one can study parts of Chaps. 8 and 9 if more calculus is desired. Minimal transition material will have to be supplied to maintain continuity of presentation. Thus the structure and contents of the book allow flexibility in the formation of a one-year course, once a background of necessary and important results has been built up. Our experience indicates that about 23 three-hour weeks are needed to cover the first six chapters.

We are indebted to McGraw-Hill for the luxury of a preliminary edition, which enabled us to class-test the material at Williams College for two years. Our students provided substantial constructive criticisms and suggestions, which we have incorporated into this edition. Special thanks go to Eileen Sprague, who typed the major part of both the preliminary edition and the final manuscript, and to Angie Giusti and Miriam Grabois, who assisted in this task. We wish to thank all those who have provided encouragement and support throughout the preparation of the book.

GEORGE F. FEEMAN
NEIL R. GRABOIS

In analytic geometry a rectangular or cartesian coordinate system is imposed on the plane to study its geometry. This coordinate system involves two perpendicular lines, called the *axes* of the coordinate system, on which units of distance are marked off. Each point in the plane is assigned a pair of numbers which locate it relative to these axes. The most significant feature of this coordinate system is that there is a one-to-one correspondence between points in the plane and ordered pairs of numbers: to each point there is assigned exactly one pair of numbers and vice versa. We begin with a restatement of this idea and then introduce operations on number pairs which correspond to geometric operations in the plane.

1.1: THE SPACE R^2

It is assumed that the reader is familiar with the real numbers and with the basic operations on these numbers. We denote the set of real numbers by R and sometimes refer to real numbers as *scalars*.

R^2 is the collection of all ordered pairs of real numbers. A point in R^2 is a pair of real numbers $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. The numbers x_1 and x_2 are called the *first* and *second coordinates*, respectively, of the point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. For reasons which will become clear in Chap. 4, we choose to write the coordinates in a column rather than in a row, as is normally done in analytic geometry. For example, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} -\frac{2}{3} \\ \sqrt{2} \end{pmatrix}$ are points in R^2. The word *ordered* means that attention must be given to the order in which the numbers are written: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ if and only if $x_1 = u_1$ and $x_2 = u_2$. The point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is different from the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. The point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in R^2, with both coordinates zero, is denoted by O and is called the *zero point* in R^2. In general, we shall use capital letters such as X and Y to denote points in R^2.

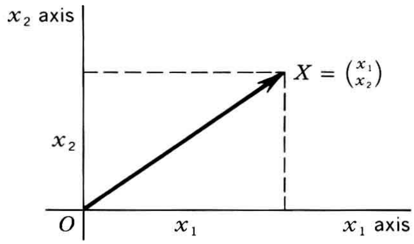

FIGURE 1 **The point $\binom{x_1}{x_2}$ and the vector $\binom{x_1}{x_2}$.**

A point $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in R^2 may be pictured in a cartesian coordinate system by an arrow which begins at the origin and has its head at the point with coordinates $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. This arrow is called a ***vector***. There is a one-to-one correspondence between points in R^2 and vectors. We denote both the point and the vector by X (see Fig. 1).

Two important operations in R^2 are the addition of points and the multiplication of a point by a real number. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. The *sum* $X + Y$ is defined to be the point in R^2 given by $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$. For example, let $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$. Then

$$X + Y = \begin{pmatrix} 2 + (-5) \\ 3 + 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

Addition can be viewed graphically as shown in Fig. 2. To add the vectors X and Y, complete a parallelogram with X and Y as the sides; the vector $X + Y$ is the diagonal of the parallelogram, which begins at the zero point O (origin).

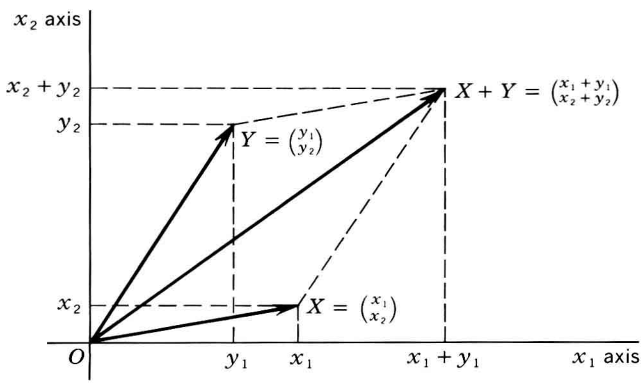

FIGURE 2 **Addition of vectors: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$.**

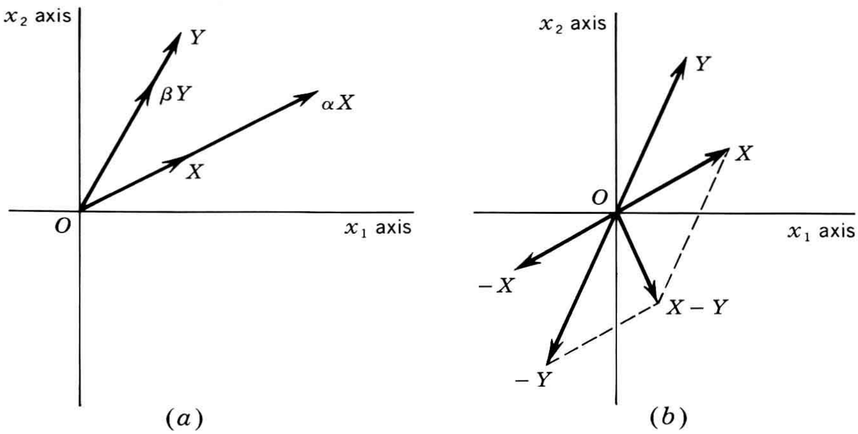

FIGURE 3 **Multiplication of vectors by scalars: (a) $0 < \beta < 1$; $\alpha > 1$; (b) $X - Y = X + (-1)Y$.**

EXERCISE

a. Let $X = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$. Find $X + Y$ and show the sum graphically.

b. Let $X = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $Y = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, and $Z = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Show that $(X + Y) + Z = X + (Y + Z)$.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and let α be any real number. The *product* αX is defined to be the point in R^2 given by $\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$. For example, if $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\alpha = 5$, then $\alpha X = 5X = \begin{pmatrix} 5(2) \\ 5(3) \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \end{pmatrix}$. Graphically, αX is a vector which lies along the line determined by the vector X and which is magnified or contracted by $|\alpha|$ (see Fig. 3a). In particular, if $\alpha = 0$, then $\alpha X = 0X = O$, for any X in R^2. Notice that the zeros in the last equation are not the same; one is a real number, the other a point in R^2. If $\alpha = -1$, we let $(-1)X = -X$. The vector $-X$ is a vector in the direction opposite to X, and its coordinates are the same as those of X except for sign.

This enables us to assign a meaning to the operation $X - Y$, namely, $X - Y = X + (-1)Y$. For example,

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 - 1 \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

(see Fig. 3b).

We may now form combinations such as $\alpha X + \beta Y$, where α and β are any real numbers and X and Y are points in R^2. For example, let $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Then

$$2X - 3Y = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} -15 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ 9 \end{pmatrix}$$

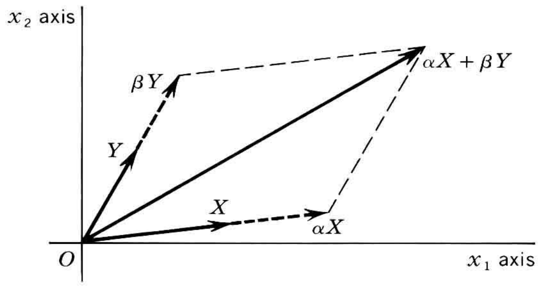

FIGURE 4 **The vector $\alpha X + \beta Y$.**

A combination $\alpha X + \beta Y$ is called a *linear combination* of the points X and Y. Graphically, $\alpha X + \beta Y$ can be shown as in Fig. 4.

The set R^2 of ordered pairs of real numbers, together with the two operations of addition and multiplication by a real number, forms the *space* R^2. Henceforth, we shall use the word *space* to mean a set together with certain operations defined on its points.

EXERCISE

Let $X = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $Y = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

a. Find $2X + 4Y$ and $3X - Y$. Show each combination graphically.

b. Find $-3X$, $Y + (-Y)$, $X + (-Y)$, and $Y + (X - Y)$.

Having shown that linear combinations can be formed, we shall prove that *any* point Z in R^2 can be expressed as a linear combination of two nonzero points X and Y such that $Y \neq \alpha X$ for any scalar α. Therefore such linear combinations completely fill the space R^2.

We shall work an example first and then prove the general result.

EXAMPLE 1.1

Let $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, and $Z = \begin{pmatrix} 17 \\ 8 \end{pmatrix}$. Find real numbers α and β such that $Z = \alpha X + \beta Y$.

Solution

Let $\begin{pmatrix} 17 \\ 8 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta\begin{pmatrix} 5 \\ 3 \end{pmatrix}$. Then $\begin{pmatrix} 17 \\ 8 \end{pmatrix} = \begin{pmatrix} \alpha + 5\beta \\ 2\alpha + 3\beta \end{pmatrix}$, so that α and β must be the solutions of the system

$$\alpha + 5\beta = 17$$
$$2\alpha + 3\beta = 8$$

Solving for α and β, we get the unique solution $\alpha = -\frac{11}{7}$, $\beta = \frac{26}{7}$. Therefore $Z = -\frac{11}{7}X + \frac{26}{7}Y$. The numbers α and β (in this case, $-\frac{11}{7}$ and $\frac{26}{7}$) are called *coordinates of Z with respect to X and Y.*

THEOREM **1.1**

Let X and Y be two nonzero points in R^2 with $Y \neq \gamma X$ for any scalar γ. Let Z be any point in R^2. Then it is possible to find a unique pair of real numbers α and β such that $Z = \alpha X + \beta Y$.

Proof

Let $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then for Z to equal $\alpha X + \beta Y$, we must have $z_1 = \alpha x_1 + \beta y_1$ and $z_2 = \alpha x_2 + \beta y_2$. Solving for α and β, we get

$$\alpha = \frac{-y_2 z_1 + y_1 z_2}{x_2 y_1 - x_1 y_2}$$

and

$$\beta = \frac{x_2 z_1 - x_1 z_2}{x_2 y_1 - y_2 x_1}$$

Since $Y \neq \gamma X$, $x_2 y_1 \neq x_1 y_2$ and the denominator $x_2 y_1 - x_1 y_2$ is not equal to zero. Therefore α and β are uniquely determined by the given equations. ■■■

The nonzero points X and Y are said to be *collinear* if $Y = \alpha X$ for some real number α. Since $O = 0X$ for any X, we say that the zero point is collinear with every point. If X and Y are noncollinear points in R^2 and $Z = \alpha X + \beta Y$, then α and β are called the *coordinates* of Z with respect to X and Y.

Theorem 1.1 has a geometric interpretation. Let X and Y be noncollinear points in R^2 and suppose that Z is any point in R^2. Construct the vectors X, Y, and Z as in Fig. 5. At the head of the vector Z draw lines parallel to the vectors X and Y, respectively. Then extend the vectors X and Y as necessary in order to complete a parallelogram. The vector Z is then the sum of αX and βY for some α and β. The numbers are unique by construction because each vector on the line determined by a nonzero vector is a unique multiple of that vector (see Prob. 17).

This theorem, which enables us to resolve any point Z in R^2 uniquely in terms of any two nonzero points X and Y such that $Y \neq \gamma X$, will be crucial to the development in Chap. 3.

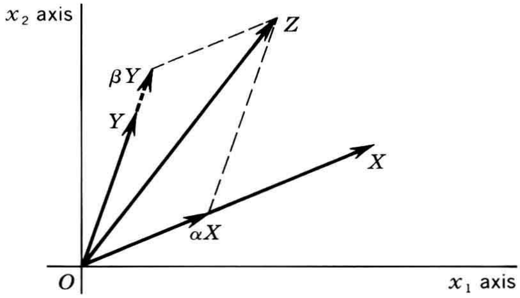

FIGURE 5 **Resolution of the vector Z into the linear combination $\alpha X + \beta Y$.**

Two observations are very much in order. First, the coordinates of a point Z with respect to a pair of points X and Y depend on the choice of X and Y. To illustrate this, let $Z = \begin{pmatrix} 17 \\ 8 \end{pmatrix}$.

a. If $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, then $\alpha = -\frac{11}{7}$ and $\beta = \frac{26}{7}$, as in Example 1.1.

b. If $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\alpha = 17$ and $\beta = 8$, since $\begin{pmatrix} 17 \\ 8 \end{pmatrix} = 17\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c. If $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $\alpha = 9$ and $\beta = 8$, since $\begin{pmatrix} 17 \\ 8 \end{pmatrix} = 9\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 8\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Once X and Y are specified in accordance with the hypothesis of Theorem 1.1, the scalars α and β are uniquely determined. Implicit in the theorem is the fact that the lines determined by any two noncollinear vectors may serve as axes for a coordinate system. We shall study the relationships among the coordinates of Z for different choices of point pairs X and Y in Chap. 4.

The second observation concerns the significance of the uniqueness of α and β. Let X_1, X_2, and X_3 be three points in R^2 no two of which are collinear. Let X be a fourth point in R^2, chosen arbitrarily. Then, by Theorem 1.1, there exist unique pairs of real numbers α_1, α_2, and β_1, β_2 such that

$$X = \alpha_1 X_1 + \alpha_2 X_2$$

and

$$X_3 = \beta_1 X_1 + \beta_2 X_2$$

Let γ be *any* nonzero real number. Then, by adding and subtracting γX_3 in the first equation, we get

$$X = \alpha_1 X_1 + \alpha_2 X_2 - \gamma X_3 + \gamma X_3$$

Substituting for X_3, we obtain

$$X = \alpha_1 X_1 + \alpha_2 X_2 - \gamma\beta_1 X_1 - \gamma\beta_2 X_2 + \gamma X_3$$

Hence

$$X = (\alpha_1 - \gamma\beta_1)X_1 + (\alpha_2 - \gamma\beta_2)X_2 + \gamma X_3$$

This shows that linear combinations of the *three* points X_1, X_2, and X_3 also completely fill the space R^2. But *the coordinates are not uniquely determined* since γ can be *any* real number. This is illustrated by a specific example.

EXAMPLE 1.2

Let $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $X_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. No two of these points are collinear. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$$X = x_1 X_1 + x_2 X_2$$

and

$$X_3 = X_1 + X_2$$

For γ *any* real number, we have

$$X = (x_1 - \gamma)X_1 + (x_2 - \gamma)X_2 + \gamma X_3$$

For instance, if $\gamma = 1$, then

$$\begin{aligned} X &= (x_1 - 1)X_1 + (x_2 - 1)X_2 + X_3 \\ &= (x_1 - 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_2 - 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

If $\gamma = 2$, then

$$X = (x_1 - 2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_2 - 2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Any other number substituted for γ will give a different combination, so that the resolution of X in terms of X_1, X_2, and X_3 is *not unique*.

PROBLEMS

1. Let $X = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $Y = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$. Find $4X - 5Y$ and $-2X + 7Y$.
2. Find x_1 and x_2 if $\begin{pmatrix} 4 \\ 7 \end{pmatrix} + 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ -13 \end{pmatrix}$.
3. Find x_1 and x_2 if $\begin{pmatrix} x_1 \\ -8 \end{pmatrix} - 2\begin{pmatrix} 3 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$.

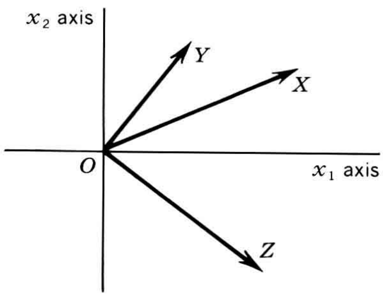

FIGURE 6

4. A factory requires 10 tons of steel and 5 tons of coal per week. Represent this demand by a point in R^2. Does it make sense to multiply this demand point by 2? How about by -1? Does it make sense to add the steel-coal demands of two factories? Of all factories?

5. Let $X = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$. Find α and β such that $Z = \alpha X + \beta Y$.

6. Let $X = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $Y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and $Z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find α and β such that $Z = \alpha X + \beta Y$.

7. Let $Z = \begin{pmatrix} 10 \\ 9 \end{pmatrix}$. Find α and β such that $Z = \alpha X + \beta Y$ for:

a. $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **b.** $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

8. Let $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $X_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

a. Find α and β such that $X = \alpha X_1 + \beta X_2$.
b. Find α and β such that $X_3 = \alpha X_1 + \beta X_2$.
c. Find two different sets of coordinates α, β, and γ such that $X = \alpha X_1 + \beta X_2 + \gamma X_3$.

9. Repeat Prob. 8 for $X = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$.

10. Let X, Y, and Z be three points in R^2. Show graphically that $(X + Y) + Z = X + (Y + Z)$.

11. Let X and Y be points in R^2 such that $X + Y = O$. Show that $X = -Y$ and that $Y = -X$.

12. Let X and Y be vectors in the plane. Show that the heads of the vectors $X + Y$, $X + 2Y$, $X + 3Y$, and $X - Y$ all lie on the same line.

13. Let the coordinates of Z with respect to $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $Y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ be -1 and 5, respectively. Find Z.

14. Let X be a nonzero point in R^2. Show that if $\alpha X = O$, then $\alpha = 0$.

15. Let α be a nonzero real number. Show that if X is a point in R^2 such that $\alpha X = O$, then $X = O$.

16. For the vectors X, Y, and Z in Fig. 6, find the coordinates of Z with respect to X and Y.

17. Let $\mathscr{L}$ be the line in the plane determined by the nonzero vector X. Show that each vector on $\mathscr{L}$ is a multiple of X and that each multiple of X is a vector on $\mathscr{L}$.
18. Let X and Y be noncollinear points in R^2. Suppose that $\alpha X + \beta Y = O$. Show that $\alpha = \beta = 0$.
19. Let X and Y be points in R^2 such that whenever $\alpha X + \beta Y = O$ both α and β must be zero. Show that X and Y are noncollinear points.
20. Let X_1 and X_2 be noncollinear points in R^2. Show that if

$$\alpha_1 X_1 + \alpha_2 X_2 = \beta_1 X_1 + \beta_2 X_2$$

then

$$\alpha_1 = \beta_1 \qquad \text{and} \qquad \alpha_2 = \beta_2$$

21. Let $X_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $X = \begin{pmatrix} -8 \\ 5 \end{pmatrix}$.
    - **a.** Find α and β such that $X = \alpha X_2 + \beta X_3$.
    - **b.** Find α and β such that $X_1 = \alpha X_2 + \beta X_3$.
    - **c.** Find two sets of coordinates α, β, and γ such that $X = \alpha X_1 + \beta X_2 + \gamma X_3$.
22. Suppose $Z = \begin{pmatrix} -8 \\ 5 \end{pmatrix}$. Find (trial and error permissible) a pair of nonzero points X and Y in R^2 such that $Z = 2X + 3Y$. Is this pair unique? If not, find a second pair.
23. Let $X = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Let $Z = X + \alpha(Y - X)$, where α is any real number. Show Z graphically for $\alpha = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1. Give a geometric description of Z for any value of α. Is it possible for Z to be the zero point O?

1.2: THE SPACE R^n

The set R^n, where n is a positive integer, is the collection of all ordered n-tuples of real numbers. A point X in R^n is then a column of n real numbers $\begin{pmatrix} x_1 \\ \cdot\ \cdot\ \cdot \\ x_n \end{pmatrix}$, the word *ordered* again meaning that two such columns are equal if and only if the corresponding real numbers are the same. For convenience, we shall frequently write (x_i) for the column $\begin{pmatrix} x_1 \\ \cdot\ \cdot\ \cdot \\ x_n \end{pmatrix}$, where x_i stands for the ith coordinate of the point. Then $(x_i) = (y_i)$ if and only if $x_i = y_i$, for $i = 1\ .\ .\ .\ , n$. The zero point O in R^n is the point with all coordinates zero, namely, $\begin{pmatrix} 0 \\ \cdot\ \cdot\ \cdot \\ 0 \end{pmatrix}$.

The set R^1 is the set of real numbers R with braces around each number. We shall normally omit both the superscript and the braces in this case. The set R^2 was introduced in the previous section. R^3 is the collection of ordered triples $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. A point $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in R^3 may be pictured by a vector, using a 3-dimensional cartesian coordi-