The Calculus of Variations and Optimal Control

An Introduction

George Leitmann

Preface

When the Tyrian princess Dido landed on the North African shore of the Mediterranean sea she was welcomed by a local chieftain. He offered her all the land that she could enclose between the shoreline and a rope of knotted cowhide. While the legend does not tell us, we may assume that Princess Dido arrived at the correct solution by stretching the rope into the shape of a circular arc and thereby maximized the area of the land upon which she was to found Carthage. This story of the founding of Carthage is apocryphal. Nonetheless it is probably the first account of a problem of the kind that inspired an entire mathematical discipline, the calculus of variations and its extensions such as the theory of optimal control.

This book is intended to present an introductory treatment of the calculus of variations in Part I and of optimal control theory in Part II. The discussion in Part I is restricted to the simplest problem of the calculus of variations. The topic is entirely classical; all of the basic theory had been developed before the turn of the century. Consequently the material comes from many sources; however, those most useful to me have been the books of Oskar Bolza and of George M. Ewing. Part II is devoted to the elementary aspects of the modern extension of the calculus of variations, the theory of optimal control of dynamical systems. Here the approach is not variational but rather geometric; it is based on a theory developed in collaboration with Austin Blaquière of the University of Paris.

This volume is the outgrowth of lecture notes for a course on the variational calculus and optimal control which has been taught at the University of California at Berkeley for over twenty years. Based on this experience, I believe that a first-year graduate student in an engineering or applied science curriculum should possess the requisite mathematical sophistication required for a reading of this text.

Over the years I have benefited greatly from fruitful discussions with many colleagues and students, too numerous to list here; they know who they are. However, two of them merit special mention. I am deeply grateful to Martin Corless and to Wolfram Stadler for their critical reading of the manuscript and for their constructive suggestions. I am also indebted to David G. Luenberger, William E. Schmitendorf, and Thomas L. Vincent for allowing me to quote from their work in Sections 13.12, 15.8, and 15.9 of the book.

George Leitmann

Symbols and Notation

Standard mathematical symbols and notation are used in this book. The most commonly used symbols are defined first. Thereafter we give the definitions of the basic notation employed in the text.

Symbols

$\triangleq$	equals by definition; denotes
$=$	equals, is equivalent to
$\neq$	does not equal; is not equivalent to
$\equiv$	equals identically; is the same as
$\not\equiv$	does not equal identically; is not the same as
$\leq$ ($\geq$)	is less (greater) than or equal to
$<$ ($>$)	is less (greater) than
$\forall$	for all, for every
$\in$	is an element (member) of; belongs to
$\notin$	is not an element (member) of; does not belong to
$\emptyset$	empty set
$\subset$	is a subset of; is contained in
$\supset$	contains
$\cup$	union
$\cap$	intersection
$\times$	Cartesian product
$\{e \mid P\}$	set of all e having property P
R^1	set of real numbers; real line
$[a, b]$	$\{x \in R^1 \mid a \leq x \leq b\}$

(a,b)	$\{x \in R^1 \mid a < x < b\}$
$(a,b]$	$\{x \in R^1 \mid a < x \leq b\}$
$[a,b)$	$\{x \in R^1 \mid a \leq x < b\}$
$\setminus$	subtraction of sets; that is, $A \setminus B \triangleq \{e \mid e \in A, e \notin B\}$
$\lvert\ \rvert$	absolute value
$\lVert\ \rVert$	Euclidean norm
inf(sup)	infimum(suprēmum)
min(max)	minimum(maximum)
sgn x	signum; that is, for $x \in R^1$, sgn $x = 1$ if $x > 0$, sgn $x = -1$ if $x < 0$
T	transpose (superscript)

Spaces

The set of all ordered n-tuples of real numbers is denoted by R^n; that is,

$$R^n \triangleq R^1 \times R^1 \times \cdots \times R^1 \qquad (n \text{ times}).$$

Thus, given an ordered n-tuple of real numbers, $x_1, x_2, \ldots, x_n$, we consider it to be a *vector* $x \in R^n$. We let all vectors be column vectors; that is,

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \ x_2 \cdots x_n]^T.$$

By endowing R^n with the *natural basis* $\{e^1, e^2, \ldots, e^n\}$, where $e^i \in R^n$ and

$$e^{iT} e^j = \delta_{ij}, \qquad \delta_{ij} \triangleq \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

we assure that R^n is a Euclidean space. In particular it follows that

$$x^T x = \sum_{i=1}^{n} x_i^2 = \lVert x \rVert^2.$$

The *closure of a set* $X \subset R^n$, denoted by $\overline{X}$, is the set together with all of its accumulation points; that is,

$$\overline{X} \triangleq X \cup \{x \mid \text{there is a sequence } x_i \in X,\ i = 1, 2, \ldots, \text{ such that } x_i \text{ converges to } x\}.$$

Functions

Given the nonempty sets X and Y,

$$f(\cdot): X \to Y$$

denotes a *function* (mapping) from the domain X into the range Y; that is, associated with $x \in X$ there is one and only one $y \in Y$. We write $y = f(x)$, and we term $f(x)$ the value of the function at x.

The scalar-valued function $f(\cdot): [a, b] \to R^1$, $[a, b] \subset R^1$, $a < b$, is of *class* C^k if and only if it and its first k derivatives are continuous on $[a, b]$. Such a function of class C^1 is also called *smooth*.

The scalar-valued function $f(\cdot): [a, b] \to R^1$, $[a, b] \subset R^1$, $a < b$, is *piecewise continuous* if and only if it is continuous on $[a, b]$ with the exception of a finite number of points of (a, b) where it possesses defined left and right limits; that is, if $f(\cdot)$ is discontinuous at $\bar{x} \in (a, b)$, then

$$f(\bar{x} - 0) \triangleq \lim_{\substack{x \to \bar{x} \\ x < \bar{x}}} f(x)$$

and

$$f(\bar{x} + 0) \triangleq \lim_{\substack{x \to \bar{x} \\ x > \bar{x}}} f(x)$$

are defined. In order to have $f(x)$ defined for all $x \in [a, b]$, we take

$$f(a) = f(a + 0),$$
$$f(b) = f(b - 0),$$

and if $f(\cdot)$ is discontinuous at $\bar{x} \in (a, b)$ we take

$$f(\bar{x}) = f(\bar{x} - 0)$$

or

$$f(\bar{x}) = f(\bar{x} + 0).$$

The function $f(\cdot): [a, b] \to R^1$, $[a, b] \subset R^1$, $a < b$, is *piecewise smooth* if and only if it is continuous and its first derivative is piecewise continuous on $[a, b]$. If the first derivative is discontinuous at $\bar{x} \in (a, b)$, then the point $(x, y) = (\bar{x}, f(\bar{x}))$ is termed a *corner* of $f(\cdot)$.

The same notation is used for a vector-valued function $f(\cdot):[a,b]\to R^n, [a,b]\subset R^1, a<b$, provided the appropriate conditions are satisfied by its components which are scalar-valued; for instance, $f(\cdot):[a,b]\to R^n$ is of class C^k if and only if the functions $f_i(\cdot):[a,b]\to R^1$, $i=1,2,\dots,n$, are of class C^k, where $f(x)\triangleq[f_1(x)\ f_2(x)\cdots f_n(x)]^T$.

The function $f(\cdot)$: $X\to R^1$, $X\subset R^m$, is of class C^k if and only if it and its partial derivatives up to and including order k are continuous on X.

Given a function $f(\cdot)$: $X\to Y$ and $Z\subset X$, the *restriction* of $f(\cdot)$ to Z, denoted by $f(\cdot)|_Z$, is the function $f(\cdot)|_Z$: $Z\to Y$ such that $f(x)|_Z=f(x)$ for all $x\in Z$.

Consider the function $o(\cdot):[a,b]\to R^n$, $[a,b]\subset R^1$, $a<0$, $b>0$, such that

$$\lim_{x\to 0}\frac{o(x)}{x}=0.$$

Every function having this property is denoted by $o(\cdot)$.

The function $f(\cdot):[a,b]\to R^n$, $[a,b]\subset R^1$, is *convex* if and only if, for every x and y in $[a,b]$ and for every $\alpha\in[0,1]$, we have

$$f(z)\le\alpha f(y)+(1-\alpha)f(x),$$

where

$$z=\alpha y+(1-\alpha)x.$$

Finally, given a function $f(\cdot)$: $X\to R^1$, $X\subset R^n$, that is differentiable at $x\in X$, we let

$$\text{grad } f(x)\triangleq\left[\frac{\partial f(x)}{\partial x_1}\ \frac{\partial f(x)}{\partial x_2}\cdots\frac{\partial f(x)}{\partial x_n}\right]^T.$$

Contents

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Part I

Calculus of Variations

1

Introduction

All of us "know" the answer to the question: What is the shape of the shortest plane curve connecting two given points? Of course it is a straight line. In mathematical terms one may pose this question as follows. Consider the family of all piecewise smooth functions

$$y(\cdot):[x_0, x_1]\to R^1, \qquad x_0 < x_1,$$

satisfying

$$y(x_0)=y_0, \qquad y(x_1)=y_1,$$

where x_0, x_1, y_0, and y_1 are prescribed. Find a function $y^*(\cdot)$ in the family defined above that yields the curve of minimum length joining points (x_0, y_0) and (x_1, y_1). For given $y(\cdot)$, the length of the curve is

$$\int_{x_0}^{x_1}\left[1+(dy/dx)^2\right]^{1/2} dx \tag{1.1}$$

so that we seek $y^*(\cdot)$ such that

$$\int_{x_0}^{x_1}\left[1+(dy^*/dx)^2\right]^{1/2} dx \le \int_{x_0}^{x_1}\left[1+(dy/dx)^2\right]^{1/2} dx \tag{1.2}$$

for all $y(\cdot)$ in the class specified above.

A less trivial and considerably more difficult problem is that of determining the thrust program which results in maximizing the flight distance or range of a rocket plane in horizontal flight; see also Exercise 3.6

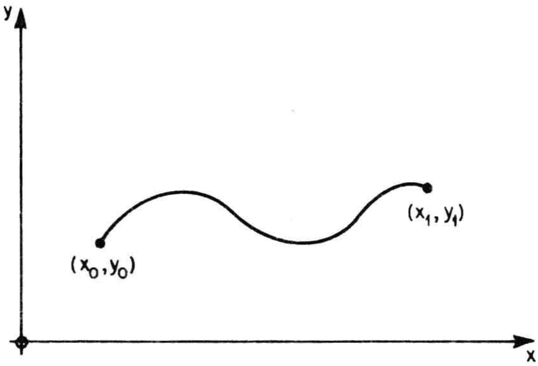

Figure 1.1. A curve connecting given points.

and Section 14.10. Let

$$
\begin{aligned}
t &= \text{time},\\
x &= \text{horizontal distance (range)},\\
v &= \text{speed},\\
m &= \text{mass},\\
T &= -c\, dm/dt = \text{thrust}, \qquad c = \text{constant} > 0,\\
L &= \text{lift},\\
D &= \text{drag}.
\end{aligned}
$$

We assume that the lift, L, is adjusted to balance the weight, mg, $g =$ constant > 0, so that the rocket moves horizontally. We assume further that the aerodynamic drag depends on the speed and lift as follows (Ref. 1.1):

$$D = Av^2 + BL^2, \qquad A \text{ and } B = \text{constants} > 0.$$

Then the equations of motion of the rocket are

$$
\begin{aligned}
\frac{dx}{dt} &= v,\\
m\frac{dv}{dt} &= c\beta - D, \qquad D = Av^2 + Bg^2m^2, \qquad (1.3)\\
\frac{dm}{dt} &= -\beta.
\end{aligned}
$$

Now, given the initial and terminal values of the speed v and mass m (and

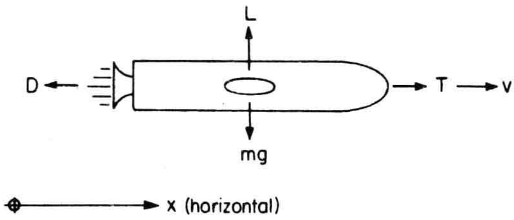

Figure 1.2. Forces acting on a rocket.

hence of the fuel consumed), we wish to determine the thrust program—that is, how T must be varied—in order to maximize the range $x(t_1)-x(0)$.

From (1.3) one obtains

$$dx=-\frac{cv}{D}\left[dm+\frac{m}{c}dv\right]$$

so that the range is

$$x(t_1)-x(0)=\int_{m_1}^{m_0}\frac{cv}{D}\left[1+\frac{m}{c}\frac{dv}{dm}\right]dm \tag{1.4}$$

where $m(0)=m_0$, $m(t_1)=m_1<m_0$, $v(0)=v_0$ and $v(t_1)=v_1$ are prescribed. We may now restate the problem more simply; namely, determine the speed v as a function of the mass m, satisfying the given end conditions and maximizing the value of the integral (1.4). From (1.3) one then has

$$T=c\beta=\frac{cD}{c+m\,dv/dm} \tag{1.5}$$

yielding the thrust program as a function of m and v, and thence of m.

These examples typify the simplest problems of the calculus of variations. In the next chapter we state such problems in a general way.