An Introduction to Calculus with Applications

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Preface

This book is intended primarily for the liberal arts student taking a calculus course for one or two semesters (or up to three quarters). The emphasis in this text is on elementary topics in calculus and certain topics from more advanced areas. These have been selected due to their applications in the fields of business, social science and biology, as well as from the physical sciences and mathematics. The topics are developed in a nonrigorous and intuitive manner, with stress being placed on the interpretation and applications of the material presented.

It is assumed that students using this text will have had courses covering the basic topics of algebra and trigonometry. However, several sections on algebra and trigonometry are included for review and reference purposes. These are the topics essential to the proper development and understanding of the calculus.

The general topics covered in the text are basic analytic geometry, differentiation and integration of algebraic and elementary transcendental functions, an introduction to partial derivatives and double integrals, basic statistics with empirical curve fitting, and expansion of functions in series.

The analytic geometry is developed primarily for use in the calculus, although numerous direct applications of curves such as the conic sections are discussed. A chapter on polar coordinates is included to demonstrate the use of a different coordinate system. Also, the empirical curve fitting is included to show how a curve may be fitted to data. In the chapter dealing with partial derivatives and double integrals, a section is devoted to solid analytic geometry. In addition to these specific topics from analytic geometry, graphical techniques and interpretations are included throughout the text. The graphs of the trigonometric, exponential, and logarithmic functions are also included.

The calculus will give the student a mathematical understanding of topics which arise in his other courses. Numerous applications from many fields are included to show the wide application of calculus. Among these are exponential growth and decay with problems from biology, economics, physics, and chemistry. Other important applications include the determination of areas and volumes of various types of geometric figures, the important physical concepts of velocity and acceleration, and marginal profit as related to business and economics.

Most instructors will probably find that not all of the material included in the text is necessary for their particular courses. Some of the topics included may be omitted without loss of continuity. Among these are some of the analytic geometry and some of the later portions of the text. These topics are included to give the instructor flexibility in developing his course to fit the needs of his students.

One of the basic features of the text is the use of more than 350 worked examples. These examples are used advantageously to clarify and illustrate the points made in the text. The author has found that these worked examples are very beneficial to the student.

There are nearly 2000 exercises. The answers to nearly all the odd-numbered exercises, including answers to graphical problems, are given at the back of the book. Also, the last section of each chapter is a set of miscellaneous exercises. These may be used either for additional problems or for review assignments.

The author wishes to acknowledge the suggestions given him by many of those who have used much of the material which has been used in this text. Among these I wish to thank the members of the Mathematics Department of Dutchess Community College. In particular, I wish to thank John Davenport of Dutchess Community College for his help in checking much of the material, including the answers. Finally, the suggestions, assistance, and cooperation of the entire staff at Cummings Publishing Company are also very deeply appreciated.

A. J. W.

Poughkeepsie, New York
January, 1972

GREEK ALPHABET

Letters	*Names*	*Letters*	*Names*	*Letters*	*Names*
Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ κ	Kappa	Σ σ	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

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7 Derivatives of the Exponential and Logarithmic Functions

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Plane Analytic Geometry

1

1-1 Introduction

During the rebirth of learning of the Renaissance, on into the seventeenth century, men made numerous discoveries regarding the world and universe in which they lived. New investigations into astronomy by Copernicus (1473–1543) and Kepler (1571–1630) led to new concepts as to the motion of the Earth and the other heavenly bodies. The work of Galileo (1564–1642), Fermat (1601–1665), and Pascal (1623–1662) was very important in advancing knowledge in basic physical science. Mathematical methods useful to these investigations were developed by Descartes (1596–1650), among others.

The furthering of knowledge in the physical sciences led to many additional unsolved problems, including those associated with the motion of the planets and stars and the motion of objects on the Earth, such as projectiles. These in turn led to mathematical problems regarding rates of change, the direction of curves, and areas bounded by curves.

In the latter part of the seventeenth century, Sir Isaac Newton (1642–1727) and Gottfried Leibnitz (1646–1716), generalizing on the works of many who came before them, independently created the calculus. By this branch of mathematics, many problems of motion, areas, and numerous other problems, many of which are apparently unrelated, may be solved.

After the development of the basic concepts, other men, such as Euler (1707–1783), Lagrange (1736–1813) and Cauchy (1789–1857), continued to further complete and clarify the calculus. Throughout this development, the impact of the calculus on science, especially physics, was enormous. Through the study of mechanics it became possible to accurately describe and predict the motions of planets and other celestial objects. Optics became a highly developed subject. In the nineteenth century the mathematical basis for electricity was established, leading to the development of electricity as a very important energy source, which has had such an impact on all phases of our culture. The advent of the calculus also had a dramatic affect on the very way men thought, in philosophy, religion, and literature.

Into the twentieth century, the mathematical developments of the previous three centuries, particularly the calculus, significantly influenced the development of our culture. Certainly the extreme acceleration of scientific developments in the twentieth century has been made possible by the mathematics developed over the past few hundred years and which is still being developed.

Today many branches of mathematics, some which seemed unimportant to applications in the recent past and others which were unknown 25 to 50 years ago, are of great importance in applied areas. Mathematics, including the calculus, is of greater importance than ever in such fields as biology, business, and the social sciences. The impact of the modern computer has been unparalleled in furthering the knowledge in all of these areas.

The calculus allows us to solve numerous problems beyond those which can be solved by the methods of algebra and trigonometry. The methods of *differential calculus*, which we shall start developing in Chapter 2, will enable us to solve problems involving the rate of change of one quantity with respect to another. One important example of a rate of change is velocity, the rate of change of distance with respect to time. Other applications are found in the examples and exercises in the text.

Another principal type of problem which we can solve with the calculus is that of finding a quantity when its rate of change is known. This is *integral calculus*, a study of which starts in Chapter 4. One important application of this is determining a population when its rate of growth is known. Integral calculus also leads to the solution of a great many other apparently unrelated problems, which include plane areas and volumes of geometric figures.

Before starting the study of calculus, we will first develop in this chapter some of the basic concepts of *analytic geometry*. This branch of mathematics deals with the relationship between algebra and geometry. The definitions and methods developed here will enable us to develop the calculus more readily. Also, many applications of analytic geometry itself are illustrated.

1-2 Rectangular coordinates

From algebra we recall that a *variable* is a quantity which may take on admissible values during a given discussion. A *constant* is a quantity which remains fixed during the discussion. Generally, letters toward the end of the alphabet are used to denote variables, and letters near the beginning of the alphabet denote constants.

One of the most valuable ways of representing an equation in two variables is by means of a graph of the equation. By using graphs we are able to obtain a "picture" of the equation, and this picture enables us to learn a great deal about the equation.

To achieve this graphical representation we use the fact that we may represent numbers by points on a line. Since two variables are involved, it is necessary to use a line for each. This is done most conveniently by placing the lines perpendicular to each other.

Thus we place one line horizontally and label it the *x-axis*. The other line we place vertically and label it the *y-axis*. The point of intersection is called the *origin*, designated by 0. This is called the *rectangular coordinate system.*

Starting from the origin, equal intervals are marked off along each of the axes, and the *integers* (the numbers 0, 1, 2, 3, . . . and their negatives) are placed at these positions. The *positive* integers are placed on the x-axis to the right of the origin, and the *negative* integers are placed to the left. On the y-axis, the positive integers are placed above the origin and the negative integers are placed below. The four parts into which the plane is divided are called *quadrants.* (See Fig. 1–1.)

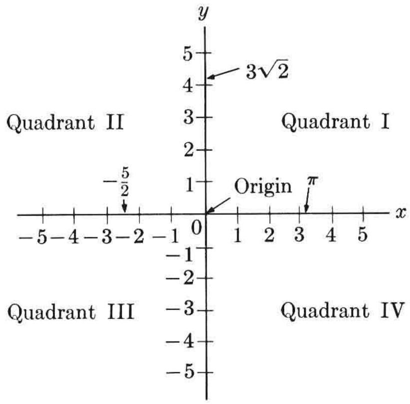

Figure 1-1

The points between the integers represent either the *rational* numbers (numbers which are represented by dividing one integer by another) or the *irrational* numbers (those numbers which cannot be represented by dividing one integer by another; π and $\sqrt{2}$ are such numbers). In this way all *real* numbers, which include the rational numbers (the integers are rational) and irrational numbers, may be represented as points on the axes. We shall use real numbers throughout this text unless otherwise noted. We will occasionally refer to and use *imaginary* numbers, which is the name given to square roots of negative numbers.

Example A. The number 7 is an integer, rational (since $7 = \frac{7}{1}$), and real (the real numbers include all the rational numbers); 3π is irrational and real; $\sqrt{5}$ is irrational and real; $\frac{1}{8}$ is rational and real; $7\sqrt{-1}$ is imaginary; $\frac{6}{3}$ is rational and real (it is an integer when we use the symbol "2" to represent it); $\pi/6$ is irrational and real; $\sqrt{-3}/2$ is imaginary.

A point P in the plane is designated by the pair of numbers (x, y). The x-value, called the *abscissa* is the perpendicular distance of P from the y-axis. The y-value, called the *ordinate,* is the perpendicular distance of P from the x-axis. The values x and y together, written as (x, y), are the *coordinates* of P.

Example B. The positions of points $P(4, 5)$, $Q(-2, 3)$, $R(-1, -5)$, $S(4, -2)$ and $T(0, 3)$ are shown in Fig. 1–2. Note that this representation allows for *one point for any pair of values* (x, y).

Example C. Where are all the points whose ordinates are 2? All such points are two units above the x-axis; thus the answer can be stated as "on a line 2 units above the x-axis." Note the dashed line in Fig. 1–2.

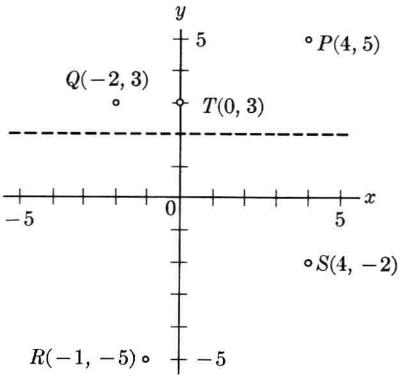

Figure 1-2

The number of units that a given number on either axis is from the origin is called the *absolute value* of that number. The absolute value of any number, except zero, is considered to be positive, and is designated by | | placed around the number.

Example D. The absolute value of 6, $|6|$, equals 6. Also, $|-7| = 7$, $|-\pi| = \pi$, $|\frac{7}{5}| = \frac{7}{5}$, $|-\sqrt{2}| = \sqrt{2}$.

On the x-axis, if a first number is to the right of a second number, the first number is said to be *greater than* the second. If the first number is to the left of the second, it is *less than* the second. On the y-axis, if one number is above a second, the first number is greater than the second. "Greater than" is designated by $>$, and "less than" is designated by $<$. These are called *signs of inequality.*

Example E.

$$6 > 3, \quad 8 > -1, \quad 5 < 9, \quad 0 > -4, \quad -2 > -4, \quad -1 < 0.$$

Exercises

In Exercises 1 and 2 designate the given numbers as integers, rational, irrational, real, or imaginary. (More than one designation may be correct.)

1. (a) $\frac{5}{4}$ (b) $-\pi$ (c) $\sqrt{-4}$ (d) $6\sqrt{-1}$
2. (a) 3 (b) $6+\sqrt{2}$ (c) $\frac{1}{3}\sqrt{7}$ (d) $\frac{7}{3}$

In Exercises 3 and 4 find the absolute values of the given numbers.

3. (a) 3 (b) $\frac{7}{2}$ (c) -4
4. (a) $-(\pi/18)$ (b) x (if $x < 0$)

In Exercises 5 and 6 insert the correct inequality signs ($>$ or $<$) between the numbers of the given pairs of numbers.

5. (a) 6 8 (b) 7 -5 (c) π -1
6. (a) -4 -3 (b) $-\sqrt{2}$ -9 (c) 0.2 0.6

In Exercises 7 through 10 plot the given points.

7. $A(2,7), B(-1,-2), C(-4,2)$
8. $A(3,\frac{1}{2}), B(-6,0), C(-\frac{5}{2},-5)$
9. $A(1, 3), B(-1, -1), C(6, 13)$ Join these points by straight-line segments. What conclusion can you make?
10. $A(-3, -2), B(-3, 5), C(3, 5)$ Join these points by straight-line segments. What conclusion can you make?

In Exercises 11 through 22 locate the indicated points, and answer the questions.

11. Where are all the points whose abscissas are 1?
12. Where are all the points whose ordinates are -3?
13. What is the abscissa of all points on the y-axis?
14. What is the ordinate of all points on the x-axis?
15. Where are all the points (x, y) for which $x > 0$?
16. Locate all points (x, y) for which $|x| < 1$.
17. Locate all points (x, y) for which $x < 0$ and $y > 1$.
18. Locate all points (x, y) for which $x > 0$ and $y < 0$.
19. In which quadrants is the ratio y/x positive?
20. In which quadrants is the ratio y/x negative?
21. Three vertices of a rectangle are $(5, 2)$, $(-1, 2)$ and $(-1, 4)$. What are the coordinates of the fourth vertex?
22. Two vertices of an equilateral triangle are $(7, 1)$ and $(2, 1)$. What is the abscissa of the third vertex?

In Exercises 23 and 24 shade in the appropriate regions for the given inequalities.

23. After three hours the bacterial population in a certain culture is less than 100,000. Using t to represent time and p to represent population, this statement may be written as "for $t > 3$, $p < 100{,}000$." Using the x-axis for values of t and the y-axis for values of p, shade in the appropriate region.
24. Less than three feet from a light source the illuminance is greater than 8 lumens/m^2. Using x for the distance and y for the illuminance, shade in the appropriate region.

1-3 The graph of an equation

The graph of an equation is the set of all points whose coordinates (x, y) satisfy the equation relating x and y. To find sets of coordinates so that we may construct the graph, we assume certain values for one of the variables, usually x, and then determine the values of the other variable by use of the equation.

Since there is no limit to the possible number of points which can be chosen, we normally select a few values of x, obtain the corresponding values of y, and plot these points. The points are then connected by a *smooth* curve (not short straight lines from one point to the next), and are normally connected from left to right.

Example A. Graph the equation $y = 3x - 5$.

If we let $x = 0$ we find that $y = -5$. Choosing another value of x, 1 for example, we find that $y = -2$. Continuing to choose a few other values of x, we tabulate these results. (It is best if the table is tabulated with increasing values of x.) Finally we join the points (Fig. 1–3).

x	y
-1	-8
0	-5
1	-2
2	1
3	4
4	7

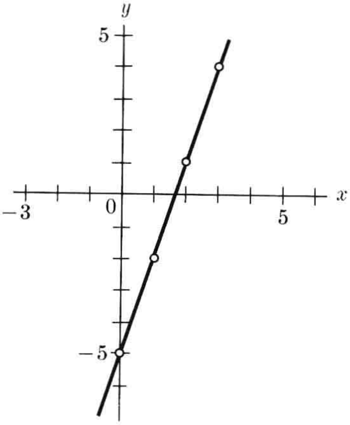

Figure 1-3

x	y
-2	-6
-1	-2
0	0
1	0
2	-2
3	-6

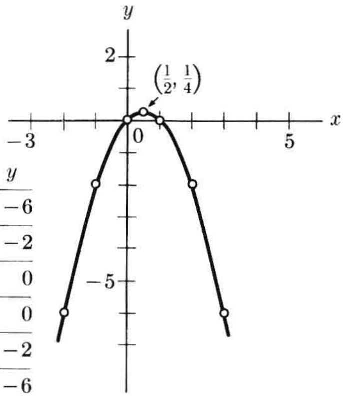

Figure 1-4

There are some special aspects of certain curves which should be noted. Each of the next three examples illustrates and explains a particular aspect to be considered when plotting the graph of an equation.

Example B. Graph the equation $y = x - x^2$.

In plotting the points shown in the table, we note that $y = 0$ for both $x = 0$ and $x = 1$. Since most common curves are smooth, any irregularities or sudden changes in the graph should be rechecked. Since there is a question as to what happens between $x = 0$ and $x = 1$, we try $x = \frac{1}{2}$. We find that $y = \frac{1}{4}$. Using this point completes the information necessary to draw a reasonable graph for the equation. (See Fig. 1–4.)

Example C. Graph the equation $y = 2x^2 - 4$.

We first tabulate the values as shown. In constructing the table we note that the same value of y appears for x-values of both $+3$ and -3. Also the same value of y appears for $+2$ and for -2, and again for $+1$ and for -1. Since $(-x)^2 = x^2$ and $(+x)^2 = x^2$, we can understand why this happens. The graph of this equation (see Fig. 1–5) is seen to be *symmetrical* to the y-axis. This can be thought of as meaning that the right half of the curve is the reflection of the left half, and conversely. As in this case, *any time $-x$ can replace x in an equation without changing the equation, its graph will be symmetrical to the y-axis.* In the same way, *if $-y$ can replace y in an equation without changing the equation, its graph is symmetrical to the x-axis.* Many of the curves to be discussed in the coming sections will demonstrate the property of symmetry.

Example D. Graph the equation $y = 1 + (1/x)$.

In finding points on this graph (see Fig. 1–6) we note that y is not defined if we try $x = 0$, since *division by zero is not defined.* Although x cannot *equal* zero, it can take on any value near zero. Thus, by choosing values of x between -1 and $+1$ we obtain the points necessary to determine the shape of the graph.

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2
2	4
3	14

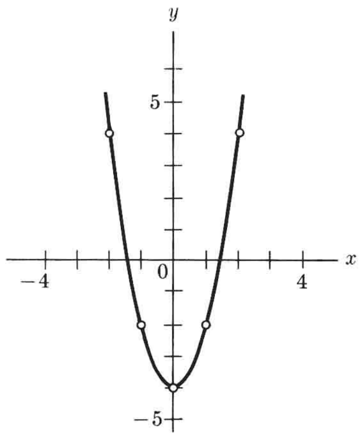

Figure 1-5

x	y
-4	3/4
-3	2/3
-2	1/2
-1	0
$-1/2$	-1
$-1/3$	-2
1/3	4
1/2	3
1	2
2	3/2
3	4/3
4	5/4

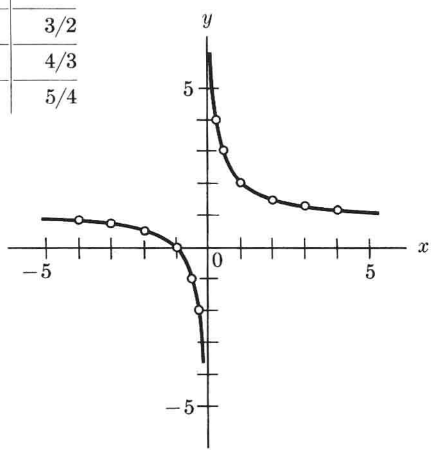

Figure 1-6