ERNST KUNZ

Introduction to Commutative Algebra and Algebraic Geometry

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Introduction to Commutative Algebra and Algebraic Geometry

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Foreword

It has been estimated that, at the present stage of our knowledge, one could give a 200 semester course on commutative algebra and algebraic geometry without ever repeating himself. So any introduction to this subject must be highly selective.

I first want to indicate what point of view guided the selection of material for this book. This introduction arose from lectures for students who had taken a basic course in algebra and could therefore be presumed to have a knowledge of linear algebra, ring and field theory, and Galois theory. The present text shouldn't require much more.

In the lectures and in this text I have undertaken with the fewest possible auxiliary means to lead up to some recent results of commutative algebra and algebraic geometry concerning the representation of algebraic varieties as intersections of the least possible number of hypersurfaces and—a closely related problem—with the most economical generation of ideals in Noetherian rings.

The question of the equations needed to describe an algebraic variety was addressed by Kronecker in 1882. In the 1940s it was chiefly Perron who was interested in this question; his discussions with Severi made the problem known and contributed to sharpening the relevent concepts. Thanks to the general progress of commutative algebra many beautiful results in this circle of questions have been obtained, mainly after the solution of Serre's problem on projective modules. Because of their relatively elementary character they are especially suitable for an introduction to commutative algebra.

If one sets the goal of leading up to these results (and some still unsolved problems), one is led into dealing with a large part of the basic concepts of commutative algebra and algebraic geometry and to proving many facts which can then serve as a basic stock for a deeper study of these subjects. Through the close linking of ring-theoretic problems with those of algebraic geometry, the rôle of commutative algebra in algebraic geometry becomes clear, and conversely the algebraic inquiries are motivated by those of geometric origin.

Since the original question is classical, we begin with classical concepts of algebraic geometry: varieties in affine or projective space. This quite naturally presents us with an opportunity to lead up to the modern generalizations (spectra, schemes) and to exhibit their utility. If the detour is not too great, we shall also pass through neighboring subjects on the way to our main goal. Yet some elementary themes of commutative algebra have been entirely neglected, among them: flat modules, completions, derivations and differentials, Hilbert polynomials and multiplicity theory. From homological algebra we use only projective resolutions and the Snake Lemma. We do not try to derive the most general known form of a proposition if to do so would seem to harm the readability of the text or if the expense seems too great. The references at the end of each chapter and the many exercises, which often contain parts of recent publications, should help the reader to become more deeply informed.

The center of gravity of this book lies more in commutative algebra than in algebraic geometry. For a continued study of algebraic geometry I recommend one of the excellent works which have recently appeared and for which this text may serve as preparation.

I shall now indicate more precisely what knowledge this book assumes.

a) The most common facts of linear and multilinear algebra for modules over commutative rings.
b) The simplest basic concepts of set-theoretic topology.
c) The basic facts of the theory of rings and ideals, including factorial rings (unique factorization domains) and the Noether isomorphism theorems for rings and modules.
d) The theory of algebraic extensions of fields, including Galois theory, as well as the basic facts about transcendence degree and transcendence bases.

Most of what is needed should come up in any introductory course on algebra, so that the book can be read in connection with such a course.

In preparing the text I have been helped by the critical remarks and many good suggestions of H. Knebl, J. Koch, J. Rung, Dr. R. Sacher, and above all Dr. R. Waldi. I have much to thank them for, as well as the Regensburg students who industriously worked on the exercises. My special thanks also goes to Miss Eva Weber for her patience in typing the manuscript.

Ernst Kunz
Regensburg, November 1978

Preface

Dr. Klaus Peters of Birkhäuser Boston has suggested that I write a few words as a Preface to the English edition of Professor Kunz's book. This book will be particularly valuable to the American student because it covers material that is not available in any other textbooks or monographs. The subject of the book is not restricted to commutative algebra developed as a pure discipline for its own sake; nor is it aimed only at algebraic geometry where the intrinsic geometry of a general n-dimensional variety plays the central role. Instead this book is developed around the vital theme that certain areas of both subjects are best understood together. This link between the two subjects, forged in the nineteenth century, built further by Krull and Zariski, remains as active as ever. It deals primarily with polynomial rings and affine algebraic geometry and with elementary and natural questions such as: What are the minimal number of elements needed to generate certain modules over polynomial rings? Great progress has been made on these questions in the last decade. In this book, the reader will find at the same time a leisurely and clear exposition of the basic definitions and results in both algebra and geometry, as well as an exposition of the important recent progress due to Quillen–Suslin, Evans–Eisenbud, Szpiro, Mohan Kumar and others. The ample exericises are another excellent feature. Professor Kunz has filled a longstanding need for an introduction to commutative algebra and algebraic geometry that emphasizes the concrete elementary nature of the objects with which both subjects began.

David Mumford

Preface to the English Edition

The English text is—except for a few minor changes—a translation of the German edition of the book *Einführung in die Kommutative Algebra und Algebraische Geometrie.* Some errors found in the original text have been removed and several passages have been better formulated. In the references the reader's attention is drawn to new findings that are in direct correlation to the contents of the book; the references were expanded accordingly.

I would like to thank all of my colleagues whose criticisms contributed toward the improvement of the text, and naturally, of course, those mathematicians who expressed their recognition of the relevance of the book. My special thanks to the translator, Mr. Michael Ackermann, for his excellent work.

Ernst Kunz
Baton Rouge, November 1981

Terminology

Throughout the book the term ring is used for a commutative ring with identity. Every ring homomorphism $R \to S$ is supposed to map the unit element of R onto the unit element of S; in particular if S/R is an extension of S over R, both S and R have the same unit element. If we say that a subset S of a ring is multiplicatively closed we always assume that $1 \in S$. If M is a module over a ring R the unit element of R operates as identity on M $(1 \cdot m = m$ for all $m \in M)$. $\mathbb{A}^n(K)$ denotes the n-dimensional affine space over the field K $(n \in \mathbb{N})$, i. e. K^n with the usual affine structure. The affine subspaces of $\mathbb{A}^n(K)$ are called "linear varieties." The same holds for the projective space $\mathbb{P}^n(K)$.

If not otherwise specified, a corollary to a proposition will contain the same assumptions as the proposition itself. If a statement is quoted, it will be given by its number if the statement is contained in the same chapter. Otherwise the number of the chapter in which the statement is found will be given first (e. g. the theorem of Quillen and Suslin, Chap. IV, 3.14). References from the list of textbooks found at the end of the book are quoted by letter, research papers are quoted by numbers. Some papers which appeared after the publication of the German edition of the book will be referred to in the text or in the footnotes.

Contents

Chapter I
Algebraic varieties

In this chapter affine algebraic varieties are introduced as the solution sets of systems of algebraic equations, and projective varieties are introduced as the solution sets in projective space of systems of algebraic equations involving only homogeneous polynomials. Hilbert's Nullstellensatz gives a necessary and sufficient condition for the solvability of a system of algebraic equations. The basic properties of varieties are discussed, and the relation to ideal theory is established. We then introduce the spectrum of a ring and the homogeneous spectrum of a graded ring and explain in what sense spectra generalize the concepts of affine and projective varieties.

1. Affine algebraic varieties

Let $\mathbb{A}^n(L)$ be n-dimensional affine space over a field L, $K \subset L$ a subfield.

Definition 1.1. A subset $V \subset \mathbb{A}^n(L)$ is called an *affine algebraic K-variety* if there are polynomials $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ such that V is the solution set of the system of equations

$$f_i(X_1, \ldots, X_n) = 0 \qquad (i = 1, \ldots, m) \tag{1}$$

in $\mathbb{A}^n(L)$. (1) is called a *system of defining equations* of V, K a field of definition of V, and L the coordinate field.

A K-variety V is also a K'-variety for any subfield $K' \subset L$ that contains all the coefficients of a system of equations defining V (e.g. if $K \subset K'$). The concept of a K-variety is invariant under affine coordinate transformations

$$X_i = \sum_{k=1}^{n} a_{ik} Y_k + b_i \qquad (i = 1, \ldots, n) \tag{2}$$

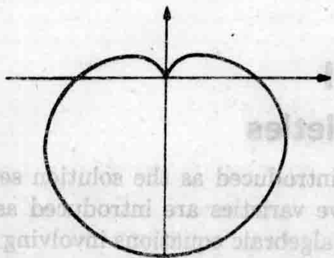
if the coefficients a_{ik} and b_i are all in K.

We first consider some

Examples 1.2.

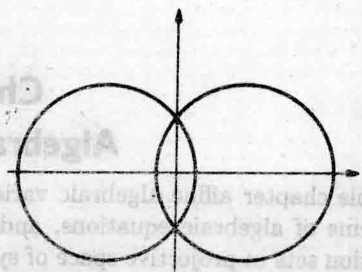
1. *Linear K-varieties.* These are the solution sets of systems of linear equations with coefficients in K. Their investigation is part of "linear algebra."
2. *K-Hypersurfaces.* These are defined by a single equation $f(X_1, \ldots, X_n) = 0$, where $f \in K[X_1, \ldots, X_n]$ is a nonconstant polynomial (cf. Figs. 3–5 and Exercise 2). For $n = 3$ hypersurfaces are also called simply "surfaces." By

Fig. 1



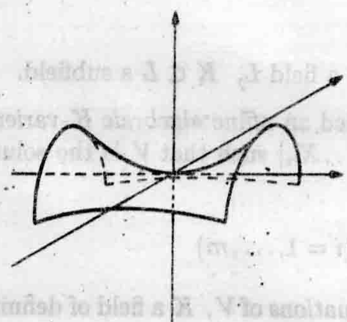
$$(X_1^2 + X_2^2 + 4X_2)^2 - 16(X_1^2 + X_2^2) = 0$$

Fig. 2



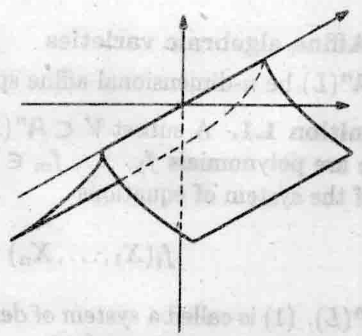
$$(X_1^2 - 9)^2 + (X_2^2 - 16)^2 + 2(X_1^2 + 9)(X_2^2 - 16) = 0$$

Fig. 3



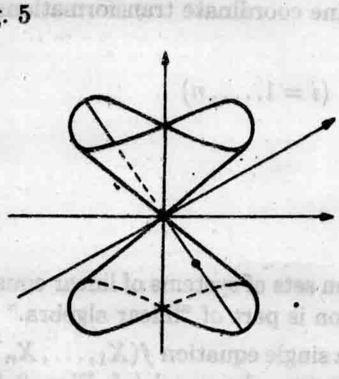
$$X_1^2 - X_2^2 - X_3 = 0$$

Fig. 4



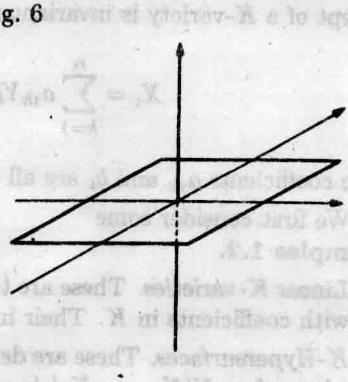
$$X_1^2 + X_3^3 = 0$$

Fig. 5



$$X_1^4 + (X_2^2 - X_1^2)X_3^2 = 0$$

Fig. 6



$$X_1X_3 = 0, \qquad X_2X_3 = 0$$

definition every affine variety is the intersection of finitely many hypersurfaces. Note that e.g. over the reals a "hypersurface" may be empty or may consist of a single point (cf. also Exercises 3 and 6). Later we shall always assume L algebraically closed; such phenomena cannot occur then.

Hypersurfaces of order 2 (quadrics) are described by equations

$$\sum_{i,k=1}^{n} a_{ik}X_iX_k + \sum_{i=1}^{n} b_iX_i + c = 0$$

(Fig. 3).

3. *Plane algebraic curves* are the hypersurfaces in $\mathbb{A}^2(L)$, i.e. the solution sets of equations $f(X_1, X_2) = 0$ with a nonconstant polynomial f in two variables (Figs. 1 and 2, Exercise 1). Such curves can be treated more simply than arbitrary varieties, and here one can often make more precise statements than in the general case. (Some textbooks that treat plane algebraic curves are: Fulton [L], Seidenberg [S], Semple-Kneebone [T], Walker [W] and Brieskorn-Knörrer [Z].)
4. *Cones.* If a variety V is defined by a system (1) with only homogeneous polynomials f_i, then it is called a K-cone with vertex at the origin. For each $x \in V$, $x \neq (0,\ldots,0)$, the whole line through x and the origin also belongs to V (Fig. 5).
5. *Quasihomogeneous varieties.* A polynomial

$$f = \sum a_{\nu_1\ldots\nu_n} X_1^{\nu_1}\cdots X_n^{\nu_n} \in K[X_1,\ldots,X_n]$$

is called quasihomogenous of type $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$ and degree $d \in \mathbb{Z}$ if $a_{\nu_1\ldots\nu_n} = 0$ for all $(\nu_1,\ldots,\nu_n)$ with $\sum_{i=1}^n \alpha_i\nu_i \neq d$. A variety is called quasihomogeneous if it is defined by a system (1) with only quasihomogeneous polynomials f_i of a fixed type α.

6. *Finite intersections and unions* of affine varieties are affine varieties (Fig. 6). It suffices to see this for two varieties. If one is defined by a system $f_i(X_1,\ldots,X_n) = 0$ $(i = 1,\ldots,m)$ and the other by a system $g_j(X_1,\ldots,X_n) = 0$ $(j = 1,\ldots,l)$, then to get the intersection one just puts the two systems together. To get the union one takes the system

$$f_i(X_1,\ldots,X_n)\cdot g_j(X_1,\ldots,X_n) = 0 \qquad (i = 1,\ldots,m;\ j = 1,\ldots,l).$$

7. *The product of two affine K-varieties.* Let $V \subset \mathbb{A}^n(L)$ be the solution set of a system $f_i(X_1,\ldots,X_n) = 0$ $(i = 1,\ldots,r)$ and $W \subset \mathbb{A}^m(L)$ the solution set of $g_j(Y_1,\ldots,Y_m) = 0$ $(j = 1,\ldots,s)$. Then the cartesian product $V \times W \subset \mathbb{A}^{n+m}(L)$ is described by the union of the two systems, the polynomials now considered as elements of $K[X_1,\ldots,X_n,Y_1,\ldots,Y_m]$.

8. *Affine algebraic groups.* For any matrix $A \in Gl(n, L)$ we can consider $(A, \det A^{-1})$ as a point of $\mathbb{A}^{n^2+1}(L)$. $Gl(n, L)$ is then identified with the hypersurface H:

$$\det(X_{ik})_{i,k=1,\ldots,n} \cdot T - 1 = 0,$$

where the X_{ik} are to be replaced by the coefficients of A and T by $\det A^{-1}$. Matrix multiplication defines a group operation on H:

$$H \times H \to H$$
$$(A, \det A^{-1}) \times (B, \det B^{-1}) \mapsto (A \cdot B, \det(AB)^{-1}).$$

Varieties which, like these, are provided with a group operation, where multiplication and inverse formation are, as with matrices, given by "algebraic relations," are called algebraic *groups.* Their theory is an independent branch of algebraic geometry (a textbook on this subject is Borel [I]).

9. *Rational points of algebraic varieties.* If $V \subset \mathbb{A}^n(L)$ is a variety and $R \subset L$ is a subring, then one is often interested in the question of whether there are points on V all of whose coordinates lie in R ("R-rational points"). For example, the Fermat Problem asks about the existence of nontrivial $\mathbb{Z}$-rational points on the "Fermat variety"

$$X_1^n + X_2^n - X_3^n = 0 \qquad (n \geq 3).$$

(A reference for such difficult questions is Lang [Q].)

We now prove some facts about affine varieties, which easily follow from the definition.

Proposition 1.3.

a) If L has infinitely many elements and $n \geq 1$, then outside any K-hypersurface in $\mathbb{A}^n(L)$ there are infinitely many points of $\mathbb{A}^n(L)$. In particular, outside any K-variety $V \subset \mathbb{A}^n(L)$ with $V \neq \mathbb{A}^n(L)$ there are infinitely many points of $\mathbb{A}^n(L)$.

b) If L is algebraically closed and $n \geq 2$, then any K-hypersurface in $\mathbb{A}^n(L)$ contains infinitely many points.

Proof.

a) Let the hypersurface be given by a nonconstant polynomial

$$F \in K[X_1, \ldots, X_n].$$

We may assume that X_n, say, actually occurs in F; we then have a representation

$$F = \varphi_0 + \varphi_1 X_n + \cdots + \varphi_t X_n^t, \tag{3}$$

with $\varphi_i \in K[X_1, \ldots, X_{n-1}]$ $(i = 0, \ldots, t)$, $t > 0$, and $\varphi_t \neq 0$. By the induction hypothesis we may assume that there is an $(x_1, \ldots, x_{n-1}) \in L^{n-1}$

with $\varphi_t(x_1,\ldots,x_{n-1}) \neq 0$. $F(x_1,\ldots,x_{n-1},X_n)$ is then a nonvanishing polynomial in $L[X_n]$. It has only finitely many zeros, but L is infinite. Hence there are infinitely many $x_n \in L$ with $F(x_1,\ldots,x_{n-1},x_n) \neq 0$.

b) Let the hypersurface be given by a polynomial F of the form (3). Then there are infinitely many $(x_1,\ldots,x_{n-1}) \in L^{n-1}$ with $\varphi_t(x_1,\ldots,x_{n-1}) \neq 0$. Since L is algebraically closed, for each of these $(x_1,\ldots,x_{n-1})$ there is an $x_n \in L$ with $F(x_1,\ldots,x_{n-1},x_n) = 0$.

Definition 1.4. For a subset $V \subset \mathbb{A}^n(L)$ the set $\mathfrak{I}(V)$ of all $F \in K[X_1,\ldots,X_n]$ with $F(x) = 0$ for all $x \in V$ is called the *ideal* of V in $K[X_1,\ldots,X_n]$ (the "vanishing ideal").

For hypersurfaces we have

Proposition 1.5.
Let L be algebraically closed and $n \geq 1$. Let $H \subset \mathbb{A}^n(L)$ be a K-hypersurface defined by an equation $F = 0$, and let $F = c \cdot F_1^{\alpha_1} \cdot \ldots \cdot F_s^{\alpha_s}$ be a decomposition of F into a product of powers of pairwise unassociated irreducible polynomials F_i $(c \in K^\times)$. Then $\mathfrak{I}(H) = (F_1 \cdot \ldots \cdot F_s)$.

Proof. Of course $F_1 \cdot \ldots \cdot F_s \in \mathfrak{I}(H)$. It suffices to show that any $G \in \mathfrak{I}(H)$ is divisible by all the F_i $(i = 1,\ldots,s)$. Suppose that, for some $i \in [1,s]$, F_i is not a divisor of G. We can think of F_i as written in the form (3). F_i and G are then (according to Gauss) also relatively prime as elements of $K(X_1,\ldots,X_{n-1})[X_n]$. Hence there are polynomials a_1, $a_2 \in K[X_1,\ldots,X_n]$ and $d \in K[X_1,\ldots,X_{n-1}]$, $d \neq 0$, such that

$$d = a_1F_i + a_2G.$$

By 1.3a) there is an $(x_1,\ldots,x_{n-1}) \in L^{n-1}$ with

$$d(x_1,\ldots,x_{n-1}) \cdot \varphi_t(x_1,\ldots,x_{n-1}) \neq 0.$$

We choose $x_n \in L$ with $F_i(x_i,\ldots,x_{n-1},x_n) = 0$. Then $(x_1,\ldots,x_n) \in H$ and so $G(x_1,\ldots,x_n) = 0$. But this is a contradiction, since $d(x_1,\ldots,x_{n-1}) \neq 0$.

Between the K-varieties $V \subset \mathbb{A}^n(L)$ and the ideals of the polynomial ring $K[X_1,\ldots,X_n]$ there is a very close connection, which is the reason that ideal theory is of great significance for algebraic geometry.

We recall the following concepts of ideal theory in a commutative ring with unity.

Definition 1.6.

1. A *system of generators* of an ideal I is a family $\{a_\lambda\}_{\lambda\in\Lambda}$ of elements $a_\lambda \in I$ such that each $a \in I$ is a linear combination of the a_λ with coefficients in R. I is called *finitely generated* if I has a finite system of generators.
2. The *ideal generated by a family* $\{a_\lambda\}_{\lambda\in\Lambda}$ of elements $a_\lambda \in R$ is the set of all linear combinations of the a_λ with coefficients in R. In the future we shall write $(\{a_\lambda\}_{\lambda\in\Lambda})$ for this ideal. By definition the empty family generates the zero ideal.

3. The *sum* $\sum_{\lambda\in\Lambda} I_\lambda$ of a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of a ring is the set of all sums $\sum_{\lambda\in\Lambda} a_\lambda$ with $a_\lambda \in I_\lambda$, $a_\lambda \neq 0$ for only finitely many λ.
4. The *product* $I_1 \cdot \ldots \cdot I_n$ of finitely many ideals $I_1, \ldots, I_n$ of a ring is the ideal generated by all the products $a_1 \cdot \ldots \cdot a_n$ with $a_j \in I_j (j = 1, \ldots, n)$. In particular, this defines the *n-th power* I^n of an ideal $I : I^n$ is the ideal generated by all the products $a_1 \cdot \ldots \cdot a_n$ $(a_i \in I)$.
5. The *radical* $\mathrm{Rad}(I)$ of an ideal I is the set of all $r \in R$ some power of which lies in I. It is easily shown that $\mathrm{Rad}(I)$ is indeed an ideal. $\mathrm{Rad}(0)$ is called the *nilradical* of R. It consists of all the nilpotent elements of R, so this set is an ideal of R. A ring R is called *reduced* if $\mathrm{Rad}(0) = (0)$. For any ring R, $R_{\mathrm{red}} := R/\mathrm{Rad}(0)$ is reduced. R_{red} is called the *reduced ring* belonging to R.
6. An ideal I of R is called a *prime* ideal if the following holds: If $a, b \in R$ and $a \cdot b \in I$, then $a \in I$ or $b \in I$. I is a prime ideal if and only if R/I is an integral domain. For an arbitrary ideal I we will call any prime ideal of R that contains I a *prime divisor* of I. A prime ideal $\mathfrak{P}$ is called a *minimal prime divisor* of I if $\mathfrak{P}' = \mathfrak{P}$ for any prime divisor $\mathfrak{P}'$ of I with $\mathfrak{P}' \subset \mathfrak{P}$. From the definition of a prime ideal it easily follows that: A prime ideal that contains the intersection (or the product) or two ideals contains one of the two ideals. Moreover, $\mathrm{Rad}(\mathfrak{P}) = \mathfrak{P}$ for any prime ideal $\mathfrak{P}$.
7. An ideal $I \neq R$ is called a *maximal ideal* of R if $I' = I$ for any ideal $I' \neq R$ with $I \subset I'$. An ideal I is maximal if and only if R/I is a field.
8. The *intersection* of a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of a ring is an ideal. The same holds for the union if the following condition is satisfied: For all $\lambda_1, \lambda_2 \in \Lambda$ there is a $\lambda \in \Lambda$ with $I_{\lambda_1}, I_{\lambda_2} \subset I_\lambda$.
9. Let S/R be an extension of rings, $I \subset R$ an ideal. The *extension ideal* of I in S is the ideal generated by I in S. It is denoted by IS. More generally, if $\varphi : R \to S$ is a homomorphism of rings, IS denotes the ideal of S generated by $\varphi(I)$.

Definition 1.7. The *zero set* in $\mathbb{A}^n(L)$ of an ideal $I \subset K[X_1, \ldots, X_n]$ is the set of all common zeros in $\mathbb{A}^n(L)$ of the polynomials in I. We denote it by $\mathfrak{V}(I)$ (the "variety of I").

Once it is proved that any ideal $I \subset K[X_1, \ldots, X_n]$ has a finite system of generators $f_1, \ldots, f_m$ (§2), it will follow that $\mathfrak{V}(I)$ is a K-variety (with defining system of equations $f_i = 0$ $(i = 1, \ldots, m)$).

For the operations $\mathfrak{I}$ and $\mathfrak{V}$ the following rules hold.

Rules 1.8.

a) $\mathfrak{I}(\mathbb{A}^n(L)) = (0)$ if L is infinite; $\mathfrak{I}(\emptyset) = (1)$.
b) For any set $V \subset \mathbb{A}^n(L)$, $\mathfrak{I}(V) = \mathrm{Rad}(\mathfrak{I}(V))$.
c) For any variety $V \subset \mathbb{A}^n(L)$, $\mathfrak{V}(\mathfrak{I}(V)) = V$.
d) For two varieties V_1, V_2, we have $V_1 \subset V_2$ if and only if $\mathfrak{I}(V_1) \supset \mathfrak{I}(V_2)$, and $V_1 \subsetneq V_2$ if and only if $\mathfrak{I}(V_1) \supsetneq \mathfrak{I}(V_2)$.

e) For two varieties V_1, V_2, we have $\mathfrak{I}(V_1 \cup V_2) = \mathfrak{I}(V_1) \cap \mathfrak{I}(V_2)$ and $V_1 \cup V_2 = \mathfrak{V}(\mathfrak{I}(V_1) \cdot \mathfrak{I}(V_2))$.

f) For any family $\{V_\lambda\}_{\lambda \in \Lambda}$ of varieties V_λ,

$$\bigcap_{\lambda \in \Lambda} V_\lambda = \mathfrak{V}(\sum_{\lambda \in \Lambda} \mathfrak{I}(V_\lambda)).$$

Proof. a), b), e), and f) easily follow from the definitions.

c) Evidently $V \subset \mathfrak{V}(\mathfrak{I}(V))$. On the other hand, if V is the zero set of the polynomials $f_1, \ldots, f_m$, then $f_1, \ldots, f_m \in \mathfrak{I}(V)$ and hence $V = \mathfrak{V}(f_1, \ldots, f_m) \supset \mathfrak{V}(\mathfrak{I}(V))$.

d) From $\mathfrak{I}(V_1) \supset \mathfrak{I}(V_2)$ it follows by c) that $V_1 = \mathfrak{V}(\mathfrak{I}(V_1)) \subset \mathfrak{V}(\mathfrak{I}(V_2)) = V_2$. The remaining statements of d) are now clear.

In particular, the rules show that $V \mapsto \mathfrak{I}(V)$ is an injective, inclusion-reversing mapping of the set of all K-varieties $V \subset \mathbb{A}^n(L)$ into the set of all ideals I of $K[X_1, \ldots, X_n]$ with $\mathrm{Rad}(I) = I$. Hilbert's Nullstellensatz (§3) will show that this mapping is also bijective if L is algebraically closed. Once it is shown that any ideal in $K[X_1, \ldots, X_n]$ is finitely generated, it will follow from 1.8f) that the intersection of an arbitrary family of K-varieties in $\mathbb{A}^n(L)$ is a K-variety.

Definition 1.9. A K-variety V is called *irreducible* if the following holds: If $V = V_1 \cup V_2$ with K-varieties V_1, V_2, then $V = V_1$ or $V = V_2$.

Fig. 6 shows an example of a reducible variety. The concept of irreducibility depends in general on the field of definition K; for example, the solution set in $\mathbb{C}$ of the equation $X^2 + 1 = 0$ is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$.

Proposition 1.10. A K-variety $V \subset \mathbb{A}^n(L)$ is irreducible if and only if its ideal $\mathfrak{I}(V)$ is prime.

Proof. Let V be irreducible and let $f_1, f_2 \in K[X_1, \ldots, X_n]$ be polynomials with $f_1 \cdot f_2 \in \mathfrak{I}(V)$. For $H_i := \mathfrak{V}(f_i)$ $(i = 1, 2)$ we then have $V = (V \cap H_1) \cup (V \cap H_2)$ and so $V = V \cap H_1$ or $V = V \cap H_2$. From $V \subset H_1$ or $V \subset H_2$ it then follows that $f_1 \in \mathfrak{I}(V)$ or $f_2 \in \mathfrak{I}(V)$; i.e. $\mathfrak{I}(V)$ is prime.

Now let $\mathfrak{I}(V)$ be prime. Suppose there are K-varieties V_1, V_2 with $V = V_1 \cup V_2, V \neq V_i$ $(i = 1, 2)$. By 1.8 we have $\mathfrak{I}(V) = \mathfrak{I}(V_1) \cap \mathfrak{I}(V_2)$ and $\mathfrak{I}(V) \neq \mathfrak{I}(V_i)$ $(i = 1, 2)$. Then there are polynomials $f_i \in \mathfrak{I}(V_i), f_i \notin \mathfrak{I}(V)$ $(i = 1, 2)$. But, since $f_1 \cdot f_2 \in \mathfrak{I}(V_1) \cap \mathfrak{I}(V_2)$, we have reached a contradiction.

In the following statements let L be algebraically closed.

Corollary 1.11. A K-hypersurface $H \subset \mathbb{A}^n(L)$ is irreducible if and only if it is the zero set of an irreducible polynomial $F \in K[X_1, \ldots, X_n]$.

Namely, the principal ideal $\mathfrak{I}(H)$ (cf. 1.5) is prime if and only if it is generated by an irreducible polynomial.

Corollary 1.12. A K-hypersurface can be represented in the form

$$H = H_1 \cup \cdots \cup H_s \qquad (H_i \neq H_j \text{ for } i \neq j)$$

with irreducible K-hypersurfaces H_i. This representation is unique (up to its ordering).

Proof. Let $\mathfrak{I}(H) = (F_1 \cdot \ldots \cdot F_s)$ as in 1.5 and $H_i := \mathfrak{V}(F_i)$. Then $H = H_1 \cup \cdots \cup H_s$ ($H_i \neq H_j$ for $i \neq j$) and by 1.11 the H_i are irreducible hypersurfaces. If $H = H'_1 \cup \cdots \cup H'_t$ is any such representation, where $\mathfrak{I}(H'_j) = (G_j)$ with $G_j \in K[X_1, \ldots, X_n]$ ($j = 1, \ldots, t$), then $\mathfrak{I}(H) = (F_1 \cdot \ldots \cdot F_s) = (G_1 \cdot \ldots \cdot G_t)$ and hence $F_1 \cdot \ldots \cdot F_s = aG_1 \cdot \ldots \cdot G_t$ with $a \in K^\times$. By the theorem on unique factorization in $K[X_1, \ldots, X_n]$, we get $t = s$ and, with a suitable numbering, $H'_i = H_i$ ($i = 1, \ldots, s$).

The considerations of the next section will show that, just as for hypersurfaces, there is a unique decomposition of an arbitrary variety into irreducible subvarieties. This is important because many questions about varieties can be reduced to questions about irreducible varieties, and these are often easier to answer.

Exercises

1. Sketch the algebraic curves in $\mathbf{R}^2$ given by the following equations (especially in the neighborhood of their "singularities," i.e. where both partial derivatives of the defining polynomial vanish):

$$X_1^3 - X_2^2 = 0, \qquad X_1^5 + X_1^4 + X_2^2 = 0,$$
$$X_1^3 + X_1^2 - X_2^2 = 0, \qquad X_1^6 - X_1^4 + X_2^2 = 0,$$
$$X_1^3 + X_1^2 + X_2^2 = 0, \qquad (X_1^2 + X_2^2)^3 - 4X_1^2X_2^2 = 0,$$
$$X_1^4 - X_1^2 + X_2^2 = 0, \qquad X_1^n + X_2^n - 1 = 0.$$

(It is often advantageous to consider the points of intersection of the curve with the lines $X_2 = tX_1$ in order to get a "parametric representation" of the curve.)

2. Describe the following algebraic surfaces in $\mathbf{R}^3$ by comparing their intersections with the planes $X = c$ for variable $c \in \mathbf{R}$:

$$X^2 - Y^2Z = 0, \qquad (X^2 + Y^2)^3 - ZX^2Y^2 = 0,$$
$$X^2 + Y^2 + XYZ = 0, \qquad X^3 + ZX^2 - Y^2 = 0.$$

3. If the field K is *not* algebraically closed, then any K-variety $V \subset \mathbb{A}^n(K)$ can be written as the zero set of a single polynomial in $K[X_1, \ldots, X_n]$. (Hint: It suffices to show that for any $m > 0$ there is a polynomial $\phi \in K[X_1, \ldots, X_m]$ whose only zero is $(0, \ldots, 0) \in \mathbb{A}^m(K)$. If V is defined by a system of equations (1), put $\phi(f_1, \ldots, f_m) = 0$.)