NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION

Inverse Eigenvalue Problems

Theory, Algorithms, and Applications

MOODY T. CHU and
GENE H. GOLUB

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Inverse Eigenvalue Problems: Theory, Algorithms, and Applications

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To Joyce

for her love, patience, care, and prayer

For Neal

my beloved nephew, a gentle giant

PREFACE

In one of the classic books for student of linear algebra, *Finite Dimensional Vector Spaces*, Halmos (1974) wrote,

> Almost every combination of the adjectives proper, latent, characteristic, eigen and secular, with the nouns root, number and value, has been used in the literature for what we call a proper value.

This interesting comment on the nomenclature of eigenvalue echoes the enigmatic yet important role that eigenvalues play in nature. This entity of eigenvalues has been recognized under so many different names because its existence has been found in settings of widely varied disciplines. One instance, as Parlett (1998) put it, is that,

> Vibrations are everywhere, and so too are the eigenvalues associated with them.

In our fervent pursuit of the Knowledge of Nature, it often becomes necessary to first understand the spectral properties of the underlying physical system. It thus follows that considerable research effort has been expended on eigenvalue computation. The applications of this research furnish critical insight into the understanding of many vital physical systems.

An inverse eigenvalue problem, in contrast, concerns the reconstruction of a physical system from prescribed spectral data. The spectral data involved may consist of the complete or only partial information of eigenvalues or eigenvectors. It is obvious that the construction must be subject to some corporeal constraints due to, for instance, the structure or feasibility of the system. The objective of an inverse eigenvalue problem is to construct a physical system that maintains a certain specific structure as well as that given spectral property.

Inverse eigenvalue problems arise in a remarkable variety of applications, including system and control theory, geophysics, molecular spectroscopy, particle physics, structure analysis, and so on. Generally speaking, the basic goal of an inverse eigenvalue problem is to reconstruct the physical parameters of a certain system from the knowledge or desire of its dynamical behavior. Since the dynamical behavior often is governed by the underlying natural frequencies and/or normal modes, the spectral constraints are thus imposed. On the other hand, in order that the resulting model is physically realizable, additional structural constraints must also be imposed upon the construction. Depending on the application, inverse eigenvalue problems appear in many different forms. Our basic assumption in this presentation is that the underlying physical system is somehow represented in terms of matrices. The subsequent discussion therefore centers around eigenvalue problems, and particularly the inverse eigenvalue problems, for matrices.

Associated with any inverse eigenvalue problem are two fundamental questions – the theoretic issue on solvability and the practical issue on computability. Solvability concerns obtaining a necessary or a sufficient condition under which an inverse eigenvalue problem has a solution and whether a solution is unique. Computability concerns developing a procedure by which, knowing *a priori* that the given spectral data are feasible, a matrix can be constructed numerically. Both questions are difficult and challenging, and we still do not have complete answers. Additionally, except for a few cases most inverse eigenvalue problems have multiple solutions. The very hard yet important consideration of its sensitivity analysis should not be overlooked.

In this note our emphasis is to provide an overview of the vast scope of this fascinating problem. The fundamental questions, some known results, many applications, mathematical properties, a variety of numerical techniques, as well as several open problems will be discussed. We have to acknowledge that merely getting the current materials organized has been a formidable task since the beginning of this project. Each cross-section of this immense subject is in fact a major research effort itself with many variations. Theories and methods vary accordingly but sometimes share surprising and subtle similarities. We feel that it might be helpful to at least categorize the problems by some kinds of characteristics. As such, we divide the inverse eigenvalue problems into those that are attributed by parameters in Chapter 3, those that carry specific structures in Chapter 4, those that are characterized by partial information of eigenvalues and eigenvectors in Chapter 5, those that are of least squares nature in Chapter 6, those that are spectrally constrained in Chapter 7, those that are of low ranks in Chapter 8, and those that are specified by orbits of group actions in Chapter 9. No doubt such a classification will never be perfect. It does appear that, instead of setting forth a systematic theory and practical algorithms, we are proffering a problem book to readers. Though some of these chapters imbricate and refer to each other, readers might find some relief in knowing that each chapter can be rendered independently of each other.

We wish to have accomplished three goals in this treatise: First, we desire to demonstrate the breadth of areas where inverse eigenvalue problems can arise. The discipline ranges from practical engineering applications to abstract algebraic theorization. Secondly, we want to corroborate the depth of intricacy of inverse eigenvalue problems. While the setup of an inverse eigenvalue problem seems relatively easy, the solution is not straightforward. The instruments employed to solve such a problem are quite sophisticated, including techniques from orthogonal polynomials, degree theory, optimization, to differential geometry, and so on. Finally and most importantly, we want to arouse interest and encourage further research into this topic. Throughout the text, we wish to convey the message that there is much room for further study of the numerical development and theoretical understanding of this fascinating inverse problem.

This book is an accumulation of many years' research supported in part by the National Science Foundation under the grants DMS-9803759 and DMS-0073056. The book is based on a series of lectures first presented at the Istituto per Ricerche di Matematica Applicata (IRMA), Bari, Italy, in the summer of 2001 under the encouragement of Fasma Diele. The success of presenting those lectures was made possible by Roberto Peluso at the IRMA and Dario Bini at the Universitá di Pisa with the support from Il Consiglio Nazionale delle Ricerche (CNR) and the Gruppo Nazionale per il Calcolo Scientifico (GNCS) under the project "Matrici con struttura, analisi, algoritmi e applicazioni". Later in the fall of 2001 the same series was presented at the National Center of Theoretical Sciences (NCTS), Hsinchu, Taiwan, upon the invitation by Wen-Wei Lin at the Tsinhua University. At about the same time, we received a summons from Arieh Iserles to write a treatise for *Acta Numerica.* This sequence of events inadvertently kindled the fire within us to further extend and complete this project. Many other people have made various contributions to this project. We are especially indebted to Hua Dai at the Nanjing University of Aeronautics and Astronautics, Graham Gladwell at the University of Waterloo, Robert Plemmons at the Wake Forest University, Yitshak Ram at the Louisiana State University, and Shufang Xu at the Peking University, for their comments, suggestions and generous assistance. The heartfelt kindness and encouragement received from these many dear colleagues are greatly appreciated.

Moody T. Chu and Gene H. Golub
Raleigh, North Carolina and Stanford, California
October, 2004

CONTENTS

LIST OF ACRONYMS

IEP	Inverse Eigenvalue Problem
ISVP	Inverse Singular Value Problem

AIEP	Additive IEP
ECIEP	Equality Constrained IEP
ISEP	Inverse Singular/Eigenvalue Problem
JIEP	Jacobi IEP
LiPIEP	Linear Parameterized IEP
LSIEP	Least Square IEP
LSPDIEP	Least Square Partially Described IEP
MIEP	Multiplicative IEP
MQIEP	Monic Quadratic IEP
MVIEP	Multi-Variate IEP
NIEP	Nonnegative IEP
NNMF	Nonnegative Matrix Factorization Problem
NNMP	Nearest Normal Matrix Problem
PAP	Pole Assignment Problem
PDIEP	Partially Described IEP
PEIEP	IEP with Prescribed Entries
PEISVP	ISVP with Prescribed Entries
PIEP	Parameterized IEP
PISVP	Parameterized ISVP
QIEP	Quadratic IEP
RNIEP	Real-valued Nonnegative IEP
SCAP	Spectrally Constrained Approximation Problem
SHIEP	Schur-Horn IEP
SIEP	Structured IEP
SLRAP	Structured Low Rank Approximation Problem
SNIEP	Symmetric Nonnegative IEP
SQIEP	Standard Quadratic IEP
StIEP	Stochastic IEP
STISVP	Sing-Thompson ISVP
SVCAP	Singular Value Constrained Approximation Problem
ToIEP	Toeplitz IEP
UHIEP	Unitary Hessenberg IEP

LIST OF FIGURES