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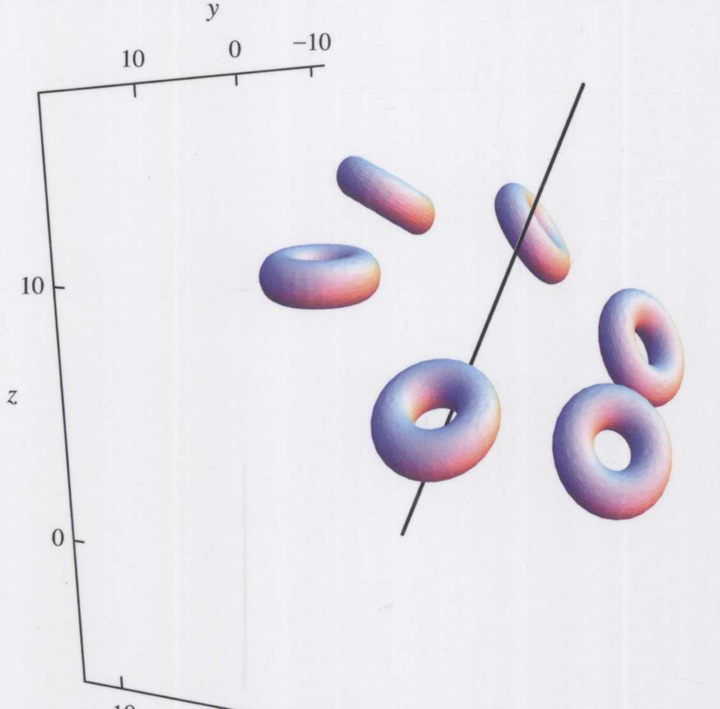
Principles of Linear Algebra with Mathematica®

Kenneth Shiskowski and Karl Frinkle



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Principles of Linear Algebra With *Mathematica*®

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Preface

This book is an attempt to cross the gap between beginning linear algebra and the computational linear algebra that one encounters more frequently in applied settings. The underlying theory behind many topics in the field of linear algebra is relatively simple to grasp; however, to actually apply this knowledge to nontrivial problems becomes computationally intensive. To do these computations by hand would be tedious at best, and many times simply unrealistic. Furthermore, attempting to solve such problems by the old pencil-and-paper method does not give the average reader any extra insight into the problem. *Mathematica*® allows readers to overcome these obstacles, giving them the power to perform complex computations that would take hours by hand, and can help to visualize many of the geometric interpretations of linear algebra topics in two and three dimensions in a very intuitive fashion. We hope that this book will challenge the reader to become proficient in both theoretical and computational aspects of linear algebra.

Overview of the Text

Chapter 1 of this book is a brief introduction to *Mathematica* and will help the reader become more comfortable with the program. This chapter focuses on the commands and programming most commonly used when studying linear algebra and its applications. *Mathematica* commands will always be in bold, with output (if any) displayed in a left-justified fashion below each *Mathematica* command. Readers can enter these commands and obtain the same results, assuming that they have entered the commands correctly. Note also that all of the images in this book were produced with *Mathematica*. The overall intent of this book is to use *Mathematica* to enhance the concepts of linear algebra, and therefore *Mathematica* is integrated into this book in a very casual manner. Where one normally explains how to perform some operation by hand in a standard text, we often simply use *Mathematica* commands to perform the same task. Thus, the reader should attempt to become familiar with the *Mathematica* syntax as quickly as possible.

At the end of each section, you will find two types of problems: "Homework problems" and "*Mathematica* problems". The former consists of strictly pen-and-pencil computation problems, inquiries into theory, and questions about concepts discussed in the section. The idea behind these problems is to ensure that the reader has an understanding of the concepts introduced and can put them to use in problems that can be worked out by hand. For example, *Mathematica* can multiply matrices together much faster than any person can and without any algebraic mistakes, so why should the reader ever perform these tasks by hand? The answer is simple: In order to fully grasp the mechanics of matrix multiplication, simple problems must be worked out by hand. This manual labor, although usually deemed tedious, is an important tool in learning reinforcement. The "*Mathematica* problems" portion of the homework typically involves problems that would take too long, or would be too computationally complex, to solve by hand. There are many problems in the "*Mathematica* problems" portion that simply ask you to verify your answers to questions from the preceding "Homework problems", implying that you can think of *Mathematica* as a "solutions manual" for a large percentage of this text. You will also notice that several sections are missing the "Homework problems" section. These sections correspond to special topics that are discussed because they can be explored in detail only with *Mathematica*.

Website and Supplemental Material

We suggest that students and instructors alike visit the book's companion Website, which can be found at either of the following addresses:

http://carmine.se.edu/kfrinkle/PrinciplesOfLinearAlgebraWithMathematica

http://people.emich.edu/kshiskows/PrinciplesOfLinearAlgebraWithMathematica

At these locations, you can download *Mathematica* notebooks, corresponding to each section's *Mathematica* commands, along with many other resources. These files can be used with the book to enable the reader to do problems or practice the material without retyping the entire *Mathematica* code. We highly suggest that all readers unfamiliar with *Mathematica* (and even those who are) read over the relevant sections of the "Introduction to *Mathematica*" notebook before they get too far into the book in order to understand the *Mathematica* code more fully. Specifically, we suggest looking at plotting/graphing material, differences between sets, lists and strings, and expressions versus functions and how *Mathematica* uses each. "Homework problems" solutions and "*Mathematica* problems" notebooks are also available for download.

Suggested Course Outlines

It would be nice if we could always cover all of the topics that we wanted to in a given course. This rarely happens, but there are obviously core topics that should be covered. Furthermore, some of the advanced topics require knowledge beyond what students in a basic linear algebra course may have. The appropriate prerequisites for this course would be trigonometry and a precalculus course in algebra. Also, a computer programming course would be helpful because we are using *Mathematica*. A year-long course in calculus would also be beneficial in regard to several topics. Here is a list of sections that require advanced knowledge:

- Section 7.2: Differentiation
- Section 7.3: Multivariable calculus
- Section 10.1: Green's theorem
- Section 10.3: Divergence theorem and double integrals
- Section 11.3: Gradients and Lagrange multipliers
- Section 12.4: Linear differential equations

We suggest that as much of the book be covered as possible, but here is the minimum suggested course outline:

Chapter 1	Sections 1.1–1.7	2 lectures
Chapter 2	Sections 2.1–2.3	4 lectures
Chapter 3	Sections 3.1–3.2	3 lectures
Chapter 5	Sections 5.1–5.6	8 lectures
Chapter 6	Sections 6.1–6.4	5 lectures
Chapter 8	Sections 8.1–8.5	7 lectures
Chapter 9	Sections 9.1–9.4	5 lectures
Chapter 11	Sections 11.1–11.3	3 lectures
Chapter 12	Sections 12.1–12.3, 12.5	3 lectures
	Total	40 lectures

On inspection of this outline, you will notice that Chapters 4, 7, and 10 have been completely omitted. Chapter 4 has interesting applications of matrix multiplication to geometry, business, finance, and curve fitting, and we highly suggest covering Sections 4.1 and 4.4. Curve fitting is covered in Section 4.2, but is covered in greater depth in Chapter 11, where pseudoinverses and the method of least-squared deviation are introduced. Chapter 7 contains applications of the information learned about vectors in Chapter 6. If you wish to cover any of the topics in Chapter 10, we highly suggest that you cover Section 7.1. Chapter 10 is a fun chapter on linear maps and how they affect geometric objects. Affine maps are included in this chapter and should be given serious consideration as a topic to cover.

Final Remarks

We hope that both students and instructors will find this book to be a unique read. Our goal was to tell a story, rather than follow the standard textbook formula of definition, theorem, example, and then repeat for 500 pages. We also hope that you really enjoy using *Mathematica*, both to explore the geometric and computational aspects of linear algebra, and to verify your pencil-and-paper work. We very much would like to hear your comments. Some of the questions we would always like answered, from both the student and the instructor, follow:

1. Were there topics that were difficult to grasp from the explanation and examples given? If so, what would you suggest that we add or change to help make comprehension easier?

2. How did you enjoy the mixture of homework and *Mathematica* problems? Did you gain anything from verifying your answers to the homework problems with *Mathematica*?

3. What were some of your favorite or least favorite sections, and why?

4. Do you feel there were important topics, integral to a first semester course in linear algebra, that were missing from this text?

5. Did embedding *Mathematica* commands and output within the actual explanation of topics help to illustrate the topics?

6. Overall, what worked the best for you in this text, and what really did not work?

It would be wonderful if this text, in its first edition, were free of errors: both grammatical and mathematical. However, no matter how many times we read and proofread this text, it is a certainty that something will be missed. We hope you contact us with any and all mistakes that you have found, along with any comments and suggestions that you may have.

Acknowledgments

First, we would like to express our thanks to Jacqueline Palmieri, Kellsee Chu, Stephen Quigley, and Susanne Steitz-Filler of John Wiley & Sons, Inc. for making the entire process, from the original proposal, to project approval, to final submission, incredibly smooth. The four of you were supportive, encouraging, very enthusiastic, and quick to respond to any questions that arose over the course of this project. We appreciate this very much. We would also like to thank the following individuals who were involved in the original peer review process:

Derek Martinez, Central New Mexico University
Dror Varolin, Stony Brook University
Gian Mario Besana, DePaul University

Chris Moretti, Southeastern Oklahoma State University
Andrew Ross, Eastern Michigan University

In addition, Chris Moretti spent a significant amount of time patiently answering many of our questions and wrote numerous manipulation procedures and standalone notebooks for the Website, some of which appear in this text. His comments and suggestions really made this book more seamlessly integrate with *Mathematica*. A special thanks also goes to Bobbi Page, who took the time to read large portions of early drafts of this book, pored over the copious copyedits, and made many invaluable suggestions. The four successive spring semester linear algebra students at Southeastern Oklahoma State University deserve a warm round of applause for being guinea pigs and error hunters. Thanks also goes out to the countless students from the many courses that Dr. Shiskowski has taught at Eastern Michigan University. We would also like to thank Mark Bickham, whose idea for a title to this book finally made both authors happy. Thanks again to everyone who was involved in this project, at any point, at any time. If we forgot to add your name this time around, perhaps you will make it into the second edition.

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With all textbooks, one should attempt to be consistent with notation, not only within the text, but also within the field of mathematics on which it is based. For the most part, we have done this.

Table of Symbols and Notation	
$\mathbf{B}$, $\mathbf{K_1}$, $\mathbf{Q}$	Bold capital letters designate sets of objects, usually vectors, or a field
$\mathbb{R}$, $\mathbb{C}$	Real and complex numbers, respectively
$\mathbb{R}^n$, $\mathbb{C}^n$	n-tuples of real and complex numbers, respectively
$\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$	$m \times n$ matrices with real and complex entries
$\mathbb{S}$, $\mathbb{T}$, $\mathbb{R}$	Math script capital letters denote vector spaces
$\dim(\mathbb{S})$	Dimension of a vector space $\mathbb{S}$
u, x e_k	Lowercase letters are designated as scalars
$\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{e_k}$	Lowercase letters with arrows over them are vectors, or column matrices
$\langle 1, 2, -1\rangle$, $\langle x_1, x_2\rangle$	Vectors expressed in component form
$\overrightarrow{u} \cdot \overrightarrow{y}$	Dot product of two vectors
$\overrightarrow{u} \times \overrightarrow{y}$	Cross product of two vectors
$\text{proj}_{\overrightarrow{v}}(\overrightarrow{w})$	Projection of $\overrightarrow{w}$ onto $\overrightarrow{v}$
$\text{comp}_{\overrightarrow{v}}(\overrightarrow{w})$	Component of $\overrightarrow{w}$ onto $\overrightarrow{v}$
A, C, X	Single capital letters represent matrices
AB, AX	Matrix multiplication has no symbol, two matrices in sequence implies multiplication
A^T, A^{-1}	Transpose, inverse of a matrix
$(A\|B)$	Augmented matrix, with A on the left, B on the right
$A_{2,3}$, $B_{j,k}$	Entries of a matrix are indexed by row,column
$\det(A)$, $\text{adj}(A)$	Determinant, adjoint of a matrix
$p(A)$	Pseudoinverse of a matrix
$T : \mathbb{R} \to \mathbb{S}$	Linear map from vector space $\mathbb{R}$ to vector space $\mathbb{S}$
$\text{Ker}(T)$, $\text{Im}(T)$	The kernel, image of a linear map T
$\int_a^b f(u)\, du$	Integral of $f(u)$ with respect to u on $[a, b]$
$\prod_{j=1}^{n} x_j$	The product $x_1 x_2 \cdots x_n$
$\sum_{j=1}^{n} x_j$	The sum $x_1 + x_2 + \cdots + x_n$
$\nabla D(\overrightarrow{a})$	Gradient of vector-valued function D
$\frac{dF}{dx}$, $\frac{\partial F}{\partial x}$	Derivative, partial derivative, of F with respect to x

Contents

Chapter 1

An Introduction to *Mathematica*

1.1 The Very Basics

Mathematica is an extremely powerful mathematical software package (or computer algebra system) that incorporates text editing, mathematical computation, and programming as well as 2D and 3D graphics capabilities. You can literally write a complete mathematics textbook using only *Mathematica* where your book includes all of the text and graphics in one smoothly flowing document. If you have never or only slightly used *Mathematica* before, then it will take some effort to learn how it works—believe me that it is well worth the time expended for the ability to do mathematically almost anything you can dream of that a computer might be able to do for you. In this introduction to *Mathematica*, you will see only a fraction of its capabilities, but hopefully enough to get you well on your way in doing 2D and 3D graphics, solving of equations, defining and using functions, lists and matrices, along with some basic mathematical programming.

This chapter discusses the fundamentals of using *Mathematica* for the novice user. If you are already familiar with *Mathematica*, you may wish to skip this chapter, although we warn you that to do so would be at your own risk. The new user of *Mathematica* will find it quite difficult in the beginning, but with practice and patience, you will master all of the basics and in time come to enjoy using *Mathematica*.

Mathematica files are called *notebooks*, and in a notebook you can place text along with input commands and their associated outputs which can be literally anything such as graphics, tables or lists, and functions. You can group the material in a notebook into different types of cells that are indicated on the right side of the notebook by brackets. At the top of the notebook you will see

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the tab *Palettes* and under it is the *Writing Assistant*, which will allow you to create new cells and/or modify cells. You can use *Writing Assistant* to change the font, color, and size of the text in your cells and you can also do this using the *Format* tab at the top of the notebook. The word processing capabilities of *Mathematica* are very similar to those of *Microsoft Word* with Ctrl+C as copy and Ctrl+V as paste, Ctrl+X as delete and copy, which can be pasted elsewhere.

The commands Alt+9 and Alt+7 will create *Input* and *Text* cells, respectively, after a horizontal line break between cells. Almost all *Mathematica* cells are *Input* or *Text* cells, or *Output* cells that are created when you activate an *Input* cell. *Input* and *Output* cells are normally in pairs with *Output* second directly following its *Input*. You can also create a new cell after a line break by typing in some text where you can control the type of cell you are writing in by using the menu which is open at the upper left of the screen next to the *Save* (or disk) icon.

Each section or chapter of a notebook file in *Mathematica* should be created as a section where the first cell of the section is a *Subtitle* cell that can be created by placing a horizontal bar between or just after a cell and then choosing *Subtitle* from the pulldown menu at the very top left of the lower ruler at the top of your screen. In order to create a subsection of this section (or chapter), do the same as just described but choose the *Subsection* from this menu. If you have not already used the *Window* tab to insert the *Toolbar* in your notebook, then please do so now. With the *Toolbar* in place, you can now change the type of cell you are in by using the pulldown menu at its far left. The *Ruler* can also be inserted into your notebook if you want it from the *Window* tab. Note that for those of us who like our text in a larger style, *Window* also has a *Magnification* feature that is quite handy.

If you wish to delete a cell (use Ctrl+X) or modify its entire contents in some way, then click on the cell tag or bracket on the right and then carry out the desired operation using *Writing Assistant* or the tabs at the top of the screen or simply Ctrl+X for a complete deletion. In text, in order to create a new paragraph in a cell, use Enter. To do the same in an *Input* cell where commands are placed, use Enter as well. If you wish to split a cell, then use Shift+Enter with the cursor at the location of the split. In using *Writing Assistant* or any pulldown tab, if you click on the triangles on the left you will open or close one of the sections inside this tab. Note that text paragraphs are not necessarily indented automatically, so you must indent them yourself manually if you want this to happen.

If you wish to close a group of cells and see only the first cell of the group (which should be the title cell of the group), then double-click on the far-right bracket for the group. You will then see a cell bracket with an arrow to the right of the cell bracket of the title or first cell of the group. If you double-click on this arrow, then you can open all the cells of this group. The copy Ctrl+C and paste Ctrl+V features of *Mathematica* are the same as those of *Microsoft*

Word and other software. If you wish to change the size, font, or other feature of a collection of cells, select one of them by clicking on its bracket and then hold the Ctrl key down while you select the rest of the cells—now go to the *Format* tab or other location and carry out your change.

It is strongly recommended that you save (use Ctrl+S) your work constantly since, like all software, *Mathematica* can glitch, which could cause you to loose some or all of your material. You should have backup copies of all of your work on a separate computer or flash drive since from our own personal experience, we know that unfortunate problems can occur.

If you are using *Mathematica* to do homework problems, it is strongly suggested that you place each problem in a single group of cells with the first cell as the title of the problem. This will make it much easier to organize your work both for yourself and the instructor who may read your material. After you have finished working in a particular *Mathematica* notebook, it is also recommended that you delete all of your output from the file unless it will take too long to recompute it. Most, if not all, of the size of a *Mathematica* file will be due to graphics, especially 3D graphics, and such files can become very large and consequently take *Mathematica* quite a while to open or save, and at such times an error can occur. Under the tab *Cell*, you have the command *Delete All Output*, which removes all output from the entire file—you might use this periodically while working in a notebook in order to shorten the file. When you reopen a notebook where all output has been deleted, you can reconstruct it all by going to the tab *Evaluation* and using the command *Evaluate Cells*; the *Input* cells will then be evaluated from the first one of the notebook to the last one.

If a *Mathematica* calculation is taking too long and/or you notice that there is an error in the input, then, in order to terminate the calculation, you should go to the *Evaluation* tab at the top of the screen and select *Abort Evaluation.* This should immediately halt the calculation in its tracks unless *Mathematica* is stuck in some enormous loop and cannot find its way out—then your only alternative might be to use Ctrl+Alt+Delete and/or turn your computer off, that is, gently pull the plug on the machine, while apologizing to it.

Beware of using capital letters to define a quantity in *Mathematica* as it might already be a built-in command name that you cannot override with something else. You should also avoid using the capital letters C, D and N for any kind of variable or name in *Mathematica* as they are also command names. The commands **Clear** and **ClearAll** will undefine a quantity that you have named. If you use the command **Exit**[], it should clear everything from memory that you have defined and *Mathematica* has produced as output by quitting the *Mathematica* kernel, which is the core of *Mathematica.*

In order to define or name a quantity in *Mathematica*, you must first decide on an appropriate name that cannot be a previously used name or *Mathematica* command name, nor should it be a common variable name like x, y, and z, which you might use in equations or functions/expressions as a variable symbol.

You can never use the same symbol or name in *Mathematica* for more than one thing. Once you decide on a name such as *TrialName*, then in an *Input* cell say **TrialName** = (or :=) where, after the equal sign, you must give the expression that is the definition of *TrialName*. In *Mathematica*, an equal sign = is used for definitions, while a double equal sign == is used in equations. You can use **!=** for not equals. The := is often used for defining functions since it suppresses output and evaluation of the named quantity.

If you are using a *Mathematica* command, but have forgotten how to use it, then place the cursor in the middle of the command name and hit the F1 key to have *Mathematica* bring up the *Help* file for this command name. You can also go directly to *Help* and type in the command name yourself, especially if you have forgotten its correct spelling. Don't forget that every command name in *Mathematica* has its first letter capitalized.

If you place a semicolon (;) at the end of a named input, then *Mathematica* will not give any associated output even though it internally carried out your command and stored it to the name given it. This feature can be useful when the output would be very long and you do not need to see it all displayed, only have it computed and/or stored.

Mathematica can use standard mathematical notation for powers k^2 and fractions $\frac{a}{b}$ where Ctrl+6 after the base k is typed will give a power location and Ctrl+/ after a numerator is typed will create a fraction and placement for the denominator—both keys must be used simultaneously. If you use these keys after a space, then *Mathematica* creates blank shells $\square^{\square}$ and $\frac{\square}{\square}$ for the appropriate quantities to be inserted.

Finally, if you are a novice or beginner at using *Mathematica*, then besides this introductory material there are many videos on *YouTube* that explain most of the basic features of *Mathematica*. It is strongly suggested that you seek these out and hopefully will find a few useful ones for doing what you are interested in. *Mathematica* itself has tutorials that you should consider using if you find them useful.

1.2 Basic Arithmetic

In this section, we will start to use *Mathematica* to do some basic arithmetic and algebra computations. In the arithmetic, which is done first, we will add and multiply, factor positive integers into products of powers of primes, find the greatest common divisor (GCD) and the least common multiple (LCM) of two positive integers, and more. In the algebra, we will factor polynomials, divide one polynomial into another to get their quotient and remainder, solve for the roots of a polynomial and also solve equations for their unknowns, and perform other algebraic operations.

In order to create an *Input* cell where you can do your calculations, go to the tab *Palettes* and bring up *Classroom Assistant*. Now click on the tab *Create*