Implementing the K-8 Curriculum and Evaluation Standards

Readings from the "Arithmetic Teacher"

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Implementing the K–8 Curriculum and Evaluation Standards

Readings from the *Arithmetic Teacher*

Edited by

Thomas E. Rowan
Montgomery County
Public Schools
Rockville, Maryland

and

Lorna J. Morrow
North York Board of Education
North York, Ontario

National Council
of Teachers
of Mathematics

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PREFACE

In March of 1989 the National Council of Teachers of Mathematics published its *Curriculum and Evaluation Standards for School Mathematics*. The companion document, *Professional Standards for Teaching Mathematics,* was published in March of 1991. These documents were "designed to establish a broad framework to guide reform in school mathematics for the next decade" (from the preface of each document). We are now about one-third of the way through that decade. The NCTM recognizes that although the *Standards* documents establish goals toward which mathematics teachers can move, their message must be interpreted and expanded in documents that give direct support to classroom teachers.

During the 1989–90 and 1990–91 school years the *Arithmetic Teacher* carried a series of articles on implementing the K–8 curriculum standards. These articles addressed each of the K–8 curriculum standards by providing additional interpretations and offering suggestions for implementation. The articles were very well received by *Arithmetic Teacher* readers. This publication compiles those articles into a single document that can be used as a reference for ideas that may help bring the *Curriculum and Evaluation Standards* to life in classrooms. Each article is published as it appeared in the *Arithmetic Teacher,* with minor editing in a few instances. At the end of this publication is an annotated bibliography of additional articles that have been published in the *Arithmetic Teacher* and that include ideas for implementing the *Curriculum and Evaluation Standards.* We have organized the articles and the references in the Bibliography into groups of related readings rather than follow the chronological sequence in which they originally appeared. We believe that this arrangement will facilitate the use of the compilation and minimize the amount of repetition that might otherwise have been necessary in the Bibliography. We have used the following groupings:

1. "Themes That Cut across Mathematics" (problem solving, communication, reasoning, connections, assessment)
2. "Number" (number sense, operations, estimation, fractions, and decimals)
3. "Space and Dimension" (geometry, spatial sense, and measurement)
4. "Data Collection and Interpretation" (statistics and probability)
5. "Patterns, Relations, Functions, and Algebra"

In addition to the articles that have appeared in the *Arithmetic Teacher,* other publications have supported the implementation of the *Curriculum and Evaluation Standards.* Notable among those being produced by NCTM are the booklets appearing in the *Curriculum and Evaluation Standards for School Mathematics* Addenda Series. The following is a list of all K–8 booklets that will be available in the complete series:

- Seven grade-level books, kindergarten through sixth grade
- *Geometry and Spatial Sense*
- *Making Sense of Data*
- *Number Sense and Operations*
- *Patterns*
- *Dealing with Data and Chance*
- *Developing Number Sense in the Middle Grades*

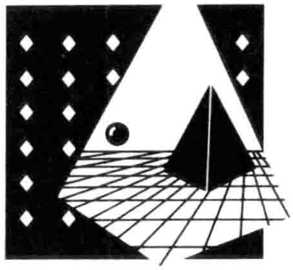
- *Geometry in the Middle Grades*
- *Measurement in the Middle Grades*
- *Patterns and Functions*
- *Understanding Rational Numbers and Proportions*

We believe that as the *Curriculum and Evaluation Standards* comes to life in classrooms, the mathematics education of children will reach new levels of success. We wish to express our special thanks to the authors of the articles that make up this compilation. Their work makes this document possible.

CONTENTS

1

Themes That Cut across Mathematics



The Vision of Problem Solving in the *Standards*

Problem solving has been espoused as a goal in mathematics education since the late 1970s, with focused attention arising from NCTM's *An Agenda for Action* (1980). But problem solving should be more than a slogan offered for its appeal and widespread acceptance. It should be a cornerstone of mathematics curriculum and instruction, fostering the development of mathematical knowledge and a chance to apply and connect previously constructed mathematical understandings. This perception of problem solving is presented in the *Curriculum and Evaluation Standards for School Mathematics* (*Standards*) (NCTM 1989, 23, 75). See **table 1.** Indeed, as noted in the *Standards,* "students need to work on problems that may take hours, days, and even weeks to solve. Although some may be relatively simple exercises to be accomplished independently, others should involve small groups or an entire class working cooperatively" (NCTM 1989, 6).

TABLE 1

Problem-solving Standards for K–4 and 5–8

K–4 standard 1

In grades K–4, the study of mathematics should emphasize problem solving so that students can—

- use problem-solving approaches to investigate and understand mathematical content;
- formulate problems from everyday and mathematical situations;
- develop and apply strategies to solve a wide variety of problems;
- verify and interpret results with respect to the original problem;
- acquire confidence in using mathematics meaningfully.

5–8 standard 1

In grades 5–8, the mathematics curriculum should include numerous and varied experiences with problem solving as a method of inquiry and application so that students can—

- use problem-solving approaches to investigate and understand mathematical content;
- formulate problems from situations within and outside mathematics;
- develop and apply a variety of strategies to solve problems, with emphasis on multistep and nonroutine problems;
- verify and interpret results with respect to the original problem situation;
- generalize solutions and strategies to new problem situations;
- acquire confidence in using mathematics meaningfully.

A statement such as that quoted above may conjure feelings of unease in many mathematics teachers. Given the already crowded curriculum in mathematics, how and when can a teacher include long-term problem-solving activities while still meeting all the other expectations of the school curriculum? The key is the approach taken to problem-solving instruction. Problem solving should not be an isolated strand or topic in the already crowded curriculum; indeed, as envisioned in the *Standards,* problem solving should pervade a mathematics program.

Edited by **Thomas E. Rowan**
Montgomery County Public Schools
Rockville, MD 20850

Prepared by **Patricia F. Campbell** *and* **Honi J. Bamberger**
University of Maryland at College Park
College Park, MD 20742-1175

The Editorial Panel welcomes readers' responses to this article or to any aspect of the Standards *for consideration for publication as an article or as a letter in "Readers' Dialogue."*

Mathematics as Problem Solving

Almost twelve years ago, Hatfield (1978) examined rationales for problem-solving instruction. More recently, Schroeder and Lester (1989) reexamined this pioneering work in light of the *Standards*. Both of these references define and distinguish among "(1) teaching *about* problem solving, (2) teaching *for* problem solving, and (3) teaching *via* problem solving" (Schroeder and Lester 1989, 32). It is useful to consider problem-solving instruction as characterized in these documents.

Teaching about problem solving refers to instruction that focuses on strategies for solving problems. For example, students may be taught first to consider the meaning of a problem and then to plan an approach to solve the problem. They are usually taught specific techniques, such as drawing a diagram, that they can use in their plan. In addition to actually solving problems, this instructional approach encourages students to think back on their solution process and to evaluate their actions as well as their solution.

Teaching for problem solving focuses on applications. This approach uses real-life problems as a setting in which students can apply and practice recently taught concepts and skills. When narrowly defined, this instructional model delays problem solving until after the introduction of a topic or computational skill and then presents a sample problem to illustrate the targeted method. Students are subsequently given similar problems to practice. The intent is for students not only to learn a host of algorithms but also to apply their understandings in a variety of contexts.

Teaching via problem solving also values applications, but not simply as a reasonable setting for using already defined mathematics. Rather, this approach uses a problem as a means of learning new mathematical ideas and for connecting new and already constructed mathematical notions. A problem can be used to initiate study of a mathematical topic, to examine one or more mathematical relationships, or to investigate mathematical ideas further. Students come to learn concepts, connect ideas, and develop skills as they solve carefully constructed problems that embody essential aspects of the mathematical content being studied.

What is problem solving as envisioned in the *Standards?* It is not simply instruction for problem solving or about problem solving. It is students actively involved in constructing mathematics through problem solving; it is cooperation and questioning as students acquire, relate, and apply new mathematical knowledge. Problem solving is a setting for communicating mathematical ideas, a context for investigating relationships, and a catalyst for connecting mathematical concepts and skills. For young pupils, problem solving should come out of everyday experience. As the students mature, problems should involve real-world settings and mathematical investigations. Problem solving is not a solitary activity, nor is it a singular strand in the mathematics curriculum.

Activities

Developmentally appropriate problems can create connections between the informal understandings that a child brings to school and the formal knowledge outlined by the mathematics curriculum (Fennema, Carpenter, and Peterson, in press). These connections create understanding in mathematics (Hiebert and Lefevre 1986).

Teachers of primary-grade pupils have numerous opportunities each day in which to generate interesting, real-life problem-solving experiences. Attendance patterns, style of shoe (tie, buckle, self-stick, or slip-on), temperature, and weather patterns are all everyday events that furnish situations for problem solving. Field trips are another source of problem-solving events.

Measurement and Seriation (Primary Level)

Some kindergarten and first-grade classes take field trips, such as a trip to a pumpkin patch in the fall. Such trips present an opportunity for comparing, measuring, predicting, and patterning, as well as for communicating mathematical ideas. Prior to a trip to the pumpkin patch, the pupils are told that they may select only one pumpkin each and that it must be one that they can carry back themselves.

How did you decide?

After returning to the classroom, the teacher asks the pupils to describe why they decided to pick their particular pumpkin. As the pupils respond, the teacher focuses their attention on the mathematics involved by reiterating and recording such words as *tallest, fattest, heaviest,* or *roundest*. The teacher also asks a pupil to point out aspects of his or her pumpkin to verify that it is indeed tall or fat or round.

How can we sort?

Once attributes of the pumpkins have been clarified through discussion, the pupils are asked to sort their pumpkins according to a given characteristic. Because pumpkins come in many sizes and shapes, the teacher guides the pupils to consider each other's perceptions, noting that the pumpkins can be sorted in more than one way. The key is to draw out the pupils' thinking, to encourage them to explain why a particular pumpkin may be assigned to a particular pile.

How can we order?

Seriation activities are done when the pupils order the pumpkins by approximate weight, height, or circumference. During this activity, the teacher's role is one of question poser and idea promoter. The pupils discuss approaches and decide together on the way in which to order the pumpkins, as well as on a way to record their data. The pumpkins are then reordered and recorded according to a different attribute. The teacher raises questions to facilitate consideration of the relationship between attributes and to encourage use of the recorded data. Is the tallest pumpkin the fattest pumpkin? If one pumpkin is fatter than another pumpkin, could it also be taller? Does it have to be taller? If one pumpkin is heavier than another pumpkin, does it have to be taller?

What do you think?

Pumpkin cleaning is another interesting mathematics activity involving measurement and cooperative grouping. This activity requires pumpkins of different sizes with their tops removed (one for each group of pupils), lots of newspapers, two or more spoons for each group, one pan balance for each group, wooden blocks, and one bowl for each group. First, the pupils are asked to predict which pumpkin will have the most "insides" to scrape out and why. They then discuss how they could find out if their prediction is close. This discussion leads to consideration of approaches for measuring the size of the pumpkins as a way of estimating and then as a way of measuring the quantity of their "insides." Measures of weight or capacity can be used for the insides. Each group of pupils scoops out the insides of their pumpkin, transports the "insides" to the bowl, and uses nonstandard units of capacity or weight, such as wooden cubes, for comparison. The scale is used for weight. The teacher then focuses the pupils' attention on devising a means of recording their data, both the size of the pumpkins and the capacity or weight of the "insides." Follow-up questions with the whole class should lead the pupils to evaluate their predictions and to relate the weight to the size of the pumpkin and to the measurement of the capacity of the pile of "insides."

Place Value (Intermediate Level)

Students' understanding of place value is a critical aspect of number sense that influences their ability to estimate and apply any of the four arithmetic operations. Although many useful manipulatives exist to introduce multidigit numbers to students, they are often used in an artificial setting that fosters the use of rules. Money, specifically pennies, dimes, and dollars, furnishes a concrete representation of place value that is meaningful to students, draws on their informal knowledge of base-ten relationships, and fosters exploration.

How many ways?

Teachers can use play money or actual currency in this activity. For either manipulative, a fixed amount of money is counted out to each small group of two to three students. The students are told how much money they are receiving in terms of the number of dimes and pennies. They verify that amount and are told that at the end of the activity they will be expected to count out that amount of money and return it to the teacher. Initially, this time also is used to verify that the students know the identity and value of pennies and dimes. Then ask the students to determine how many ways they could create $1.00 using only pennies, only dimes, or both. Some children may decide to make a chart; others may make piles of coins worth $1.00; others may realize that they could make more piles if they had more coins and so may ask to use other manipulatives to simulate money. A frequent question during this activity is, "Do I have all the ways?" Avoid directly answering this question by posing another question, "How could you find out if you have all the ways?" Subsequent full-class discussion should focus on the methods that the students used. The teacher then records the combinations derived by the groups (see **table 2**). Follow-up questions include the following:

TABLE 2

Number of dimes	Number of pennies
0	100
1	90
2	80
3	70
4	60
5	50
6	40
7	30
8	20
9	10
10	0

> What patterns do you see?
>
> What must be the sum of the values of the dimes and pennies in each row?
>
> What do all the penny amounts have in common? Why?

In this way students can draw out for themselves the ten-to-one relationship between pennies and dimes rather than listen to a rule.

What coins do I have?

This activity causes students to use and strengthen their knowledge of place-value relationships as they solve interesting problems. Students should have coins to manipulate as needed. The teacher should pose such problems as these to the class:

> "I have six coins. I have the same number of dimes as pennies. How much money do I have?"
>
> "I have six coins. Some are pennies and some are dimes. Altogether I have forty-two cents. Which coins do I have?"
>
> "I have seven coins. Three are pennies and the rest are dimes. How much money do I have?"
>
> "The bank contains pennies and dimes. I have eight coins taken from the bank. How much money could I have if I put two coins back in the bank?"

Students are expected to explain their answers and their reasoning in a full-class discussion, using either the coins or charts or verbal strings of logic. Continue sharing responses to underscore that many differing, yet acceptable, ways exist to solve these problems.

Spatial Reasoning and Logical Thinking (Intermediate Level)

This activity is not only a problem-solving activity but also an effective diagnostic task, as it reveals much about students' approaches to a problem and their persistence. It connects a variety of geometric and spatial concepts and encourages the use of experimentation, cooperation, and communication. It fosters persistence, deductive thinking, evaluation, and record keeping. Because this activity is open ended, it permits a great deal of differentiation, depending on the students' level of ability and interest, and fosters the development of mathematical vocabulary to simplify communication. Each child needs five 2-cm cubes or five 2-cm square tiles and sheets of 2-cm grid paper for the activity.

When are arrangements different?

In a full-class discussion the teacher initially identifies a single block or tile as a *monomino*. The students are asked what they think *mono-* means and then how many different ways a monomino can be arranged. This question leads to a discussion of what is "different," namely, whether the rotation of a shape yields a different

TABLE 3

Name	Number of blocks	Number of distinct arrange- ments	Shapes
Monomino	1	1	
Domino	2	1	
Tromino	3	2	

arrangement. Assuming that rotations do not produce distinct arrangements, students quickly see that only one way exists to produce a monomino. Next ask the students to use two blocks and find all the possible ways of arranging them so that one side of one block fits exactly against one side of the other block. Again the potential of a rotation yields much discussion regarding what it means to be a different arrangement. The teacher asks for a name for the two-block; *domino* is usually offered owing to its resemblance to domino game pieces. Finally, the task is extended to a three-block configuration, prompting the production of a table (see **table 3**) and the designation of the term *tromino*. Follow with an examination of other words that have the prefix *tri-* or *tr-* and their common characteristic.

Extend to four- and five-block configurations

Once the students are familiar with this line of probing, many are eager to predict that a four-block configuration is called a "quadomino." Students generally support their reasoning by refering to quadrilaterals. Although this term is not the one applied to the four-block configuration, this line of reasoning should be applauded by such comments as "That really makes a lot of sense. I think that is a perfectly good and reasonable name for a four-block arrangement. But, for some reason, it is called a *tetromino*. What do you suppose the prefix *tetra-* or *tetr-* means?" When challenged to find all possible four-block configurations, the students question each other as to whether arrangements are distinct. The students should be permitted to explore, communicate, and cooperate prior to sharing their distinct arrangements.

The fifth and final set of configurations instinctively calls the grid paper into play as a means of recording arrangements. Further, students generally ask whether they have found all the arrangements, the twelve pentominoes. Rather than answer directly, praise the students for finding so many arrangements but ask them if they think they can find more. Encourage them to use their blocks to explore arrangements, to work together to compare results, and to tell why they think they have found all the distinct configurations. If some students finish the task while others are still working, give them an extension task using the grid paper. For example, ask them to cut out the shapes of the twelve pentominoes using the grid paper and find out how many different rectangles they can make from any three of the pentominoes. Once they feel they have located every three-piece rectangle, they can move on to rectangles made from four pentomino pieces. See "Pentominoes Revisited" by Barry Onslow in this issue.

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Communication in Mathematics

The dictionary's definition of communication includes such phrases as succeeding in conveying information, having social dealings with, making connections, and making known. Our mathematical definition of communication parallels the general definition. We want our students to convey their ideas about mathematics, deal with it in social contexts, make connections, and make known their thinking as they learn and become involved in mathematics.

The *Curriculum and Evaluation Standards for School Mathematics* (*Standards*) (NCTM 1989) identifies learning to communicate mathematically as an important goal. As we reflect on this goal and look at our efforts to furnish meaningful, student-centered experiences, we examine whether what we do each day in the classroom is taking us in the direction of better communication. We want to create an atmosphere where students are given numerous opportunities to communicate their own thinking and hear from their fellow students about alternative thoughts. This article discusses the importance of communicating mathematically and examines ways to develop this skill in the classroom.

Edited by **Thomas E. Rowan**
Montgomery County Public Schools
Rockville, MD 20850

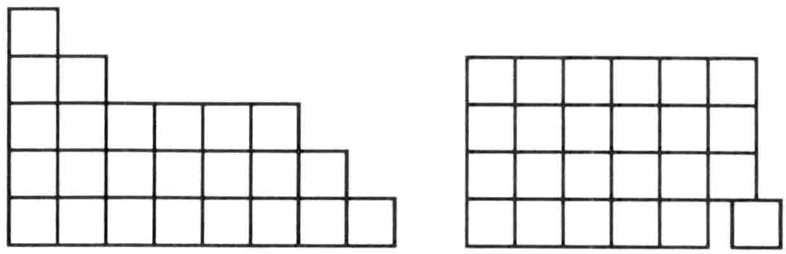
Prepared by **Judith Mumme**
University of California—
Santa Barbara
Santa Barbara, CA 93106

Nancy Shepherd
Goleta Union School District
Goleta, CA 93117

The Editorial Panel welcomes readers' responses to this article or to any aspect of the Standards *for consideration for publication as an article or as a letter in "Readers' Dialogue."*

FIGURE 1

Two dissimilar definitions of the problem produce different results.



Tiles share a complete side (even perimeter).

Tiles must touch (odd perimeter).

Communication in and about mathematics serves many functions. It helps to (1) enhance understanding, (2) establish some shared understandings, (3) empower students as learners, (4) promote a comfortable learning environment, and (5) assist the teacher in gaining insight into the students' thinking so as to guide the direction of instruction.

1. *Communication helps students enhance their understanding of mathematics.* Expressing their ideas, engaging in discussions, and listening to others help students deepen their understanding of mathematics. Listening to the thinking of others opens up new avenues for consideration, which helps students appreciate that people think in different ways and that many situations have various valid approaches.

Students construct understanding on the basis of their experiences. Communication supports their construction of knowledge by helping them clarify their thinking. This process sometimes creates disequilibrium, requiring the student to come to grips with those aspects of his or her understanding that are unclear or partially formed. For example, after seeing U.S. census data on family size, a sixth-grade class decided to collect their own data from students in their class. Working in cooperative groups, they had to decide how they would display the class data. Many developed bar graphs. One group wanted to make a circle graph but struggled with how to make the pie segments represent the right amounts. They decided to write the percentage on each segment (because they had seen this done in magazines), so they wrote a percent sign after the whole numbers from their data. Realizing that their numbers didn't add up to 100 percent, they wondered where they had gone wrong. Their dilemma prompted a rich discussion with the teacher of their ideas about the meaning of percentage. The teacher was pleased but knew that they would need many more experiences with percentage.

2. *Communication helps establish shared understandings of mathematics.* Many students fail to grasp mathematical ideas when they are presented as rules and procedures to be memorized and mastered rather than ideas to be discovered and shared. Our need to communicate with one another requires that we reach agreement on some aspects of mathematics (e.g., a numeration system, mathematical rules and conventions, definitions, etc.). By discussing and sharing ideas, students develop the need for a common language, appreciate the role of definitions, and eventually grasp the significance of discussing and clarifying assumptions.

Using square tiles, a seventh-grade class was investigating area and perimeter. Working in small groups, they were asked to find all the possible perimeters using twenty-four tiles. One group reported that they believed that only even perimeters were possible. Another group disputed this claim, indicating that they made shapes with odd perimeters. Each group explored these conflicting reports, using tiles to try to build shapes with odd or even perimeters. In a class discussion they discovered that they were defining the problem in different ways and that the results could be different, depending on how they viewed the task. Whereas some groups had decided that each tile must share a complete side with at least one other tile, as they had done with pentominoes, others had decided that any shape was permissible as long as the tiles touched (see **fig. 1**). By setting up open-ended situations, teachers help students grasp the significance of assumptions and definitions.

3. *Communication can empower students as learners.* When we ask students to talk or write about their thinking, we are telling them that we value what they have to say and communicate our confidence in their ability to think mathematically. By presenting what they think is important, students exercise greater power and control over their own learning, that is, they become empowered.

All communication in mathematics, however, does not necessarily promote empowerment. For example, in reciting the steps for doing the long-division algorithm, students can communicate that they have memorized these words but such recitation does not demonstrate their understanding of division or of when or how it is used. Contrast this exercise with students' explanations of how they could fairly share a box of raisins among five people or of the use of division to solve a problem in the real world.

Communication raises issues of control. For example, one teacher told his class that he had a friend who

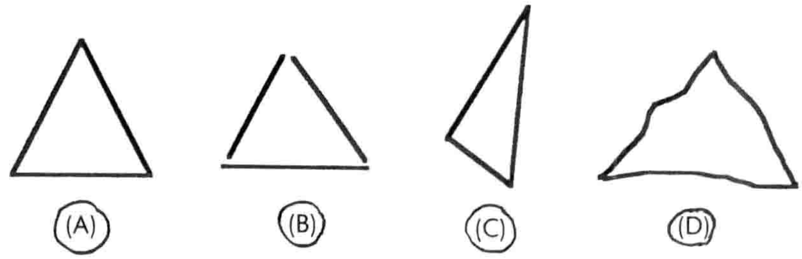
FIGURE 2

Lindsey's letter gives insight into her understanding of triangles.

Name Lindsey 6th grade

Date 12-1

1) Circle the figures that are triangles.



2) James circled figure A only. He says figures B, C, and D are not triangles. Write a letter to James telling him whether you agree with him. Tell him how you answered the question and why.

Dear James,

I do not agree with you, you do have your own appinion though and I'm not saying your wrong. I think a b c are all triangles because a looks like a perfect triangle and B even though the side are not conected is still a triangle shape, and c though it may look like half a diamond to me it is a triangle from a different angle. D looks like a peice of paper put in a blender but it can also look like a 1st graders drawing of a triangle so I think all of them look like triagles. ▲

claimed that he had collected a million pennies, which he kept in his closet. The teacher asked the students whether they thought his friend was telling the truth. An animated discussion ensued about the size of the container necessary to hold a million pennies. The teacher wanted the students to discuss what they could do to decide whether his friend was telling the truth, but the students wanted actually to experiment with their ideas—to fill small boxes and check the space 100 pennies take up, measure stacks of pennies, even weigh pennies. If teachers want to encourage mathematical thinking, they should be willing to allow students to develop their own methods of investigation.

4. *Communication promotes a comfortable environment for learning*. Talking and listening to others in small collaborative groups is an anxiety-free way to try out new ideas—to test one's thinking. Interaction with peers is enjoyable for children. Comfort and security influence their willingness to take risks in sharing their thinking.

In the example in which a circle graph was constructed, the group structure afforded the students the security to discuss ideas with each other, to admit that they didn't quite understand, and to be comfortable enough to ask the teacher, "How do we make these numbers add up to 100? Please help us. We want to know."

5. *Communication assists the teacher in gaining insight into the students' thinking*. Teachers learn a great deal about their students by listening to them explain their reasoning processes. The ability to make these explanations is an acquired skill. As with most language facility, it grows with use and practice.

The *Standards* calls for mathematics instruction to develop students' mathematical power—the ability to "explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems. This notion is based on the recognition of mathematics as more than a collection of concepts and skills to be mastered; it includes methods of investigating and reasoning, means of communication, and notions of context" (NCTM 1989, 5). Communication—talking, writing, demonstrating, drawing, and so on—is the way teachers get a glimpse of students' thinking and determine how they are developing as mathematical thinkers. Looking at how students arrived at their conclusions helps teachers gain insight into the students' conceptions and misconceptions.

FIGURE 3

Josie's description of fractions reveals some misunderstandings.

Fractions are

Josie

fractions are pieces of math that show how much amount of something is taken away or eaten.

Lets say you have a pizza cut into 24 pieces your mom eats one and you eat two that make three pieces that were taken away so now there's only 21 so you answer is $\frac{3}{21}$ you can reduce that fraction by doing this what number goes into both equally? 3. $\frac{3 \div 3}{21 \div 3} = \frac{1}{7}$ is your answer.

So fractions are just someway to get answers to pieces taken away

Figures 2–5 show some examples of students' writing that reveal some of their ideas about mathematics. In Lindsey's written response **(fig. 2)** we gain much more insight into her understanding of triangles. Josie reveals some misunderstandings she has about fractions **(fig. 3)**. Even first graders can write about their mathematical ideas **(figs. 4** and **5)**. See *A Question of Thinking* (California Assessment Program 1989) for further examples.

Fostering communication is an important aspect of mathematics instruction. Just as teachers plan the content of lessons, planning opportunities for students to engage in communication takes on increasing importance when we realize its significance in promoting learning. Communication can be fostered in several ways, including the following:

1. *Physical materials*. Using physical materials in mathematical tasks promotes communication among students by serving as a natural stimulus for talking. Students can be asked to describe a manipulative, tell what they discovered about its characteristics, what they did with it, and so on. The following is one way to use materials to encourage the development of mathematical language, spatial relationships, and communication skills. Two students pair up and place a bar-

rier, such as a book, between them so neither can see the other's desk. One student builds a shape out of patterning blocks, or some other manipulative, and describes it to the other student while he or she attempts to duplicate it. Students take turns describing and duplicating shapes. Older students can be asked to write a description as though they were giving instructions over the telephone about how to make the shape.

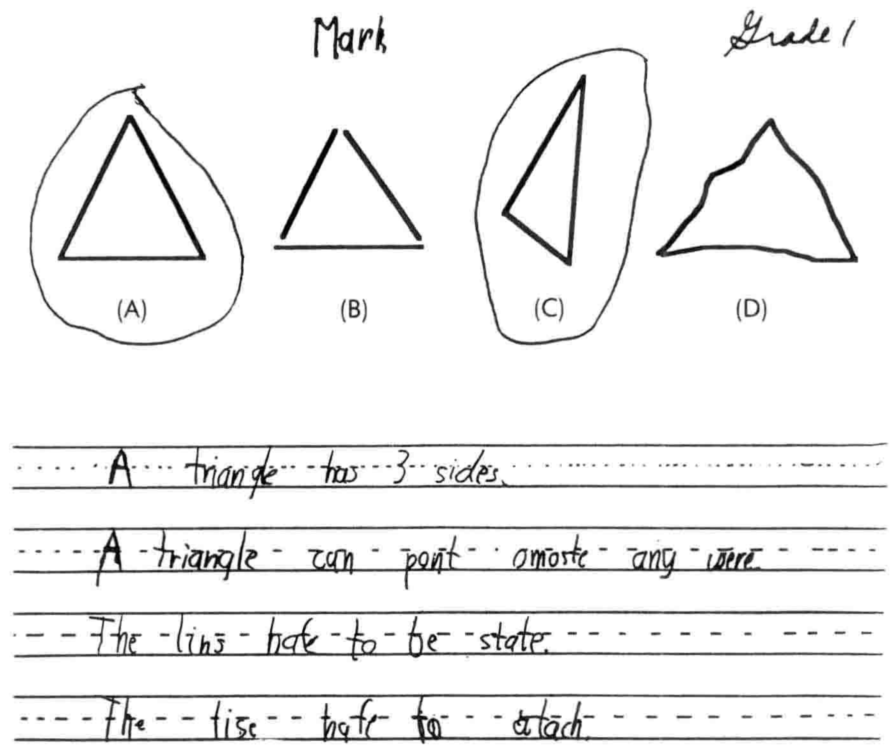
2. *Interesting and relevant topics.* Investigations, project work, and tasks that are relevant to the interests of children are ideal vehicles for promoting student-directed communication. They help students to value mathematics as a subject that is useful in their lives. Information about the students themselves affords a wealth of possibilities for mathematical communication (e.g., hobbies, pets, favorite music, etc.). Many items that students use daily spark their natural curiosity. For example, a class activity that takes advantage of young pupils' interest in money involves pupils in designing and making their own play money in different denominations. They can use the play money to play store, count by multiples (to see how much money they have), invent or solve story problems, and so on.

3. *Questions.* Open-ended questions that allow students to construct their own responses and encourage divergent or creative thinking furnish fertile areas for communication. For example, questions like "What is the biggest number?" "What is the biggest number between 0 and 1?" or "What does *straight* mean?" permit many different ideas to emerge. Allowing students to pose questions leads to some interesting discoveries. Open-ended investigations can lead older students to some powerful mathematical thinking. Students start with a problem then extend it by posing their own questions to investigate. Two examples adapted from Pirie (1987) include (1) How many 1 × 1 tiles are needed to make a border around a rectangular pool? (Some students may choose to investigate rectangles of different sizes.) (2) Choose any number. Write down all its factors, including 1 but excluding the number itself. Add these factors to get a new number. Repeat the process with this new number. Investigate.

FIGURE 4

A first grader, Mark, reveals his understanding of triangles.

Circle the figures that are triangles.



4. *Writing.* Written communication is important, and as students get used to writing, they grow to appreciate it as a part of doing mathematics. The teacher needs to help students understand why they are being asked to write—why explain it to the teacher, who already knows all that "stuff." Consequently, the purposes for writing must be made clear to students.

Initial attempts to write in mathematics can be difficult. A common response is "I don't know what to say" or "What'da ya mean?" To help students get started, make requests very specific. Asking them to write for an exact period of time, say, three to five minutes, is helpful. Have them complete a paragraph beginning with "I was trying to find . . ." or "This is what I did. . . ." Expand requests to include their conclusions and feelings. Questions like "Why do you think your answer makes sense?" or "How does this material relate to our last unit?" and "How do you feel about . . .?" reveal much more of the students' thinking.

In *A Collection of Math Lessons* (Burns 1987, 1988), the author illustrates many examples of how she has students write about their thinking. Her books are excellent resources for teachers wishing to develop writing in mathematics.

5. *Cooperative and collaborative groups.* Classroom organization and groupings are important considerations in communication. Students seated in rows are not as likely to engage in discussions as students organized in a way that allows interaction. Small collaborative groups afford opportunities to explore ideas. Whole-class discussions can be used to compare and contrast ideas from groups