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PART I: LECTURES

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Trace formula in noncommutative Geometry and the zeros of the Riemann zeta function

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Abstract

We give a spectral interpretation of the critical zeros of the Riemann zeta function, and a geometric interpretation of the explicit formulas of number theory as a trace formula on a noncommutative space. This reduces the Riemann hypothesis to the validity of the trace formula.

It is an old idea, due to Polya and Hilbert that in order to understand the location of the zeros of the Riemann zeta function, one should find a Hilbert space $\mathcal{H}$ and an operator D in $\mathcal{H}$ whose spectrum is given by the non trivial zeros of the zeta function. The hope then is that suitable selfadjointness properties of D (of $i\left(D-\frac{1}{2}\right)$ more precisely) or positivity properties of $\Delta = D(1-D)$ will be easier to handle than the original conjecture. The main reasons why this idea should be taken seriously are first the work of A. Selberg in which a suitable Laplacian Δ is related in the above way to an analogue of the zeta function, and secondly the theoretical [2, 15–17] and experimental evidence [5, 18] on the fluctuations of the spacing between consecutive zeros of zeta. The number of zeros of zeta whose imaginary part is less than $E > 0$,

$$N(E) = \# \text{ of zeros } \rho\,,\ 0 < \operatorname{Im} \rho < E \tag{1.1}$$

has an asymptotic expression [21] given by

$$N(E) = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + o(1) + N_{\rm osc}(E) \tag{1.2}$$

where the oscillatory part of this step function is

$$N_{\rm osc}(E) = \frac{1}{\pi}\ \mathrm{Im}\ \log\zeta\left(\frac{1}{2} + iE\right) \tag{1.3}$$

assuming that E is not the imaginary part of a zero and taking for the logarithm the branch which is 0 at $+\infty$.

One shows (cf. [20]) that $N_{\rm osc}(E)$ is $O(\log E)$. In the decomposition (1.2) the two terms $\langle N(E)\rangle = N(E) - N_{\rm osc}(E)$ and $N_{\rm osc}(E)$ play an independent role. The first one $\langle N(E)\rangle$ which gives the average density of zeros just comes from Stirling's formula and is perfectly controlled. The second $N_{\rm osc}(E)$ is a manifestation of the randomness of the actual location of the zeros, and to eliminate the role of the density one returns to the situation of uniform density by the transformation

$$x_j = \langle N(E_j)\rangle \quad (E_j \text{ the } j^{\rm th} \text{ imaginary part of zero of zeta})\,. \tag{1.4}$$

Thus the spacing between two consecutive x_j is now 1 in average and the only information that remains is in the statistical fluctuation. As it turns out [17, 18] these fluctuations are the same as the fluctuations of the eigenvalues of a random hermitian matrix of very large size.

H. Montgomery [17] proved (assuming RH) a weakening of the following conjecture (with $\alpha, \beta > 0$),

$$\begin{aligned} &\mathrm{Card}\,\{(i,j)\,;\, i,j \in 1,\ldots,M\,;\, x_i - x_j \in [\alpha,\beta]\} \\ &\qquad \sim M\int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du \end{aligned} \tag{1.5}$$

This law (1.5) is precisely the same as the correlation between eigenvalues of hermitian matrices of the Gaussian unitary ensemble [17]. Moreover, numerical tests due to A. Odlyzko [5, 18] have confirmed with great precision the behaviour (1.5) as well as the analogous behaviour for more than two zeros. In [15, 16], N. Katz and P. Sarnak proved an analogue of the Montgomery-Odlyzko law for zeta and L-functions of function fields over curves.

It is thus an excellent motivation to try and find a natural pair $(\mathcal{H}, D)$ where naturality should mean for instance that one should not even have to define the zeta function in order to obtain the pair (in order for instance to avoid the joke of defining $\mathcal{H}$ as the ℓ^2 space built on the zeros of zeta).

Let us first describe following [2] the direct attempt to construct the Polya-Hilbert space from quantization of a classical dynamical system. The original motivation for the theory of random matrices comes from quantum mechanics. In this theory the quantization of the classical dynamical system given by the phase space X and Hamiltonian h gives rise to a Hilbert space $\mathcal{H}$ and a selfadjoint operator H whose spectrum is the essential physical observable of the system. For complicated systems the only useful information about this spectrum is that, while the average part of the counting function,

$$N(E) = \#\text{ eigenvalues of } H \text{ in } [0, E] \tag{1.6}$$

is computed by a semiclassical approximation mainly as a volume in phase space, the oscillatory part,

$$N_{\text{osc}}(E) = N(E) - \langle N(E) \rangle \tag{1.7}$$

is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system.

In the absence of a magnetic field, i.e. for a classical Hamiltonian of the form,

$$h = \frac{1}{2m} p^2 + V(q) \tag{1.8}$$

where V is a real-valued potential on configuration space, there is a natural symmetry of classical phase space,

$$T(p, q) = (-p, q) \tag{1.9}$$

which preserves h, and entails that the correct ensemble on the random matrices is not the above GUE but rather the Gaussian orthogonal ensemble: GOE. Thus the oscillatory part $N_{\text{osc}}(E)$ behaves in the same way as for a random *real symmetric* matrix.

Of course H is just a specific operator in $\mathcal{H}$ and, in order that it behaves *generically* it is necessary (cf. [2]) that the classical Hamiltonian system (X, h) be *chaotic* with isolated *periodic orbits* whose instability exponents (i.e. the logarithm of the eigenvalues of the Poincaré return map acting on the transverse space to the orbits) are different from 0.

One can then [2] write down an asymptotic semiclassical approximation to the oscillatory function $N_{\text{osc}}(E)$

$$N_{\text{osc}}(E) = \frac{1}{\pi} \operatorname{Im} \int_0^\infty \operatorname{Trace}(H - (E + i\eta))^{-1}\, i d\eta \tag{1.10}$$

using the stationary phase approximation of the corresponding functional integral. For a system whose configuration space is 2-dimensional, this gives [2],

$$N_{\text{osc}}(E) \simeq \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2\text{sh}\left(\frac{m\lambda_p}{2}\right)} \sin(S_{\text{pm}}(E)) \tag{1.11}$$

where the γ_p are the primitive periodic orbits, the label m corresponds to the number of traversals of this orbit, while the corresponding instability exponents are $\pm\lambda_p$. The phase $S_{\text{pm}}(E)$ is up to a constant equal to $m\,E\,T_\gamma^\#$ where $T_\gamma^\#$ is the period of the primitive orbit γ_p.

The formula (1.11) gives very precious information [2] on the hypothetical "Riemann flow" whose quantization should produce the Polya-Hilbert space. The point is that the Euler product formula for the zeta function yields (cf. [2]) a similar asymptotic formula for $N_{\text{osc}}(E)$ (1.3),

$$N_{\text{osc}}(E) \simeq \frac{-1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin(m\,E\,\log p)\,. \tag{1.12}$$

Comparing (1.11) and (1.12) gives the following information,

> The periodic primitive orbits should be labelled by the prime numbers $p = 2, 3, 5, 7, \ldots$, their periods should be the $\log p$ and their instability exponents $\lambda_p = \pm \log p$.

(1.13)

Moreover, since each orbit is only counted once, the Riemann flow should not possess the symmetry T of (1.9) whose effect would be to duplicate the count of orbits. This last point excludes in particular the geodesic flows since they have the time reversal symmetry T.

However there are two important mismatches (cf. [2]) between the two formulas (1.11) and (1.12). The first one is the overall *minus sign* in front of formula (1.12), the second one is that though $2\text{sh}\left(\frac{m\lambda_p}{2}\right) \sim p^{m/2}$ when $m \to \infty$, we do not have an equality for finite values of m.

These are two fundamental difficulties and in order to overcome them we shall use the well known strategy of extending the problem of finding the hypothetical Riemann flow to the case of arbitrary *global fields.* By specializing to the function field case we shall then obtain additional precious information. The basic example of a global field is the field $\mathbb{Q}$ of rational numbers and we shall take as a conceptual definition of such fields k, the fact that they are *discrete and cocompact* in a (non discrete) locally compact semisimple Abelian ring A. As it turns out A then depends functorially on k and is called the Adele ring of k, often denoted by k_A. When the characteristic p of a global field k is > 0, the field k is the function field of a non singular algebraic curve Σ defined over a finite field $\mathbb{F}_q$ included in k as its maximal finite subfield, called the field of constants. One can then apply the ideas of algebraic geometry, first developed over $\mathbb{C}$, to the geometry of the curve Σ and obtain a geometric interpretation of the basic properties of the zeta function of k; the dictionary contains in particular the following lines

Functional equation	Riemann Roch theorem (Poincaré duality)	
Explicit formulas of number theory	Lefchetz formula for the Frobenius	(1.14)
Riemann hypothesis	Castelnuovo positivity	

Since $\mathbb{F}_q$ is not algebraically closed, the points of Σ defined over $\mathbb{F}_q$ do not suffice and one needs to consider $\bar{\Sigma}$, the points of Σ on the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$, which is obtained by adjoining to $\mathbb{F}_q$ the roots of unity of order prime to q. This set of points is a countable union of periodic orbits under the action of the Frobenius automorphism, these orbits are parametrized by the set of places of k and their periods are indeed given by the analogues of the $\log p$ of (1.13). Being a countable set it does not qualify for analogue of the Riemann flow and it only acquires an interesting structure from algebraic geometry. The minus sign which was problematic in the above discussion admits here a beautiful resolution since the analogue of the Polya-Hilbert space is given, if one replaces $\mathbb{C}$ by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers $\ell \neq p$, by the cohomology group

$$H^1_{\mathrm{et}}(\bar{\Sigma}, \mathbb{Q}_\ell) \tag{1.15}$$

which appears with an overall minus sign in the Lefchetz formula $\mathrm{Trace}/H^0 - \mathrm{Trace}/H^1 + \mathrm{Trace}/H^2$.

For the general case this suggests

The Polya-Hilbert space $\mathcal{H}$ should appear from its negative $\ominus\mathcal{H}$. (1.16)

The next thing that one learns from this excursion in characteristic $p > 0$ is that in that case one is not dealing with a flow but rather with a single transformation. In fact taking advantage of Abelian covers of Σ and of the fundamental isomorphism of class field theory one finds that the natural group that should replace $\mathbb{R}$ for the general Riemann flow is the Idele class group:

$$C_k = \mathrm{GL}_1(A)/k^* \,. \tag{1.17}$$

We can thus collect the information (1.13) (1.16) (1.17) that we have obtained so far and look for the Riemann flow as an action of C_k on an hypothetical space X.

There is a third approach to the problem of the zeros of the Riemann zeta function, due to G. Pólya [19] and M. Kac [14] and pursued further in [4, 13]. It is based on statistical mechanics and the construction of a quantum statistical system whose *partition function* is the Riemann zeta function. Such a system was naturally constructed in [4] and it does indicate using the first line of the dictionary of noncommutative Geometry what the space X should be in general:

$$X = A/k^* \tag{1.18}$$

namely the quotient of the space A of adeles, $A = k_A$ by the action of the multiplicative group k^*,

$$a \in A \ , \ q \in k^* \to aq \in A\,. \tag{1.19}$$

This space X is a noncommutative space and for instance even at the measure theory level, the corresponding von Neumann algebra,

$$R_{01} = L^\infty(A) \rtimes k^* \tag{1.20}$$

where A is endowed with its Haar measure as an additive group, is the hyperfinite factor of type II_∞.

The Idele class group C_k acts on X by

$$(j, a) \to ja \qquad \forall\, j \in C_k \ , \ a \in X \tag{1.21}$$

and it was exactly necessary to divide A by k^* so that (1.21) makes good sense.

We shall come back later to the analogy between the action of C_k on R_{01} and the action of the Galois group of the maximal Abelian extension of k.

What we shall do now is to construct the Hilbert space L^2_δ of functions on X with growth indexed by $\delta > 1$. Since X is a quotient space we shall first learn in the usual manifold case how to obtain the Hilbert space $L^2(M)$ of square integrable functions on M by working only on the universal cover $\widetilde{M}$ with the action of $\Gamma = \pi_1(M)$. Every function $f \in C_c^\infty(\widetilde{M})$ gives rise to a function $\widetilde{f}$ on M by

$$\widetilde{f}(x) = \sum_{\pi(\widetilde{x})=x} f(\widetilde{x}) \tag{1.22}$$

and all $g \in C^\infty(M)$ appear in this way. Moreover, one can write the Hilbert space inner product $\int_M \widetilde{f}_1(x)\, \widetilde{f}_2(x)\, dx$, in terms of f_1 and f_2 alone. Thus $\|\widetilde{f}\|^2 = \int \left|\sum_{\gamma\in\Gamma} f(\gamma x)\right|^2 dx$ where the integral is performed on a fundamental domain for Γ acting on $\widetilde{M}$. This formula defines a prehilbert space norm on $C_c^\infty(\widetilde{M})$ and $L^2(M)$ is just the completion of $C_c^\infty(\widetilde{M})$ for that norm. Note that any function of the form $f - f_\gamma$ has vanishing norm and hence disappears in the process of completion. In our case of $X = A/k^*$ we thus need to define the analogous norm on the Schwartz space $\mathcal{S}(A)$ of functions on A. Since 0 is fixed by the action of k^* the expression $\sum_{\gamma\in k^*} f(\gamma x)$ does not make sense for $x = 0$ unless we require that $f(0) = 0$. Moreover, when $|x| \to 0$, the above sums approximate, as Riemann sums, the product of $|x|^{-1}$ by $\int f\,dx$ for the additive Haar measure, thus we also require $\int f\,dx = 0$. We can now define the Hilbert space $L^2_\delta(X)_0$ as the completion of

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A)\ ;\ f(0) = 0\ ,\ \int f\,dx = 0\} \tag{1.23}$$

for the norm $\|\ \|_\delta$ given by

$$\|f\|^2_\delta = \int \Big|\sum_{q\in k^*} f(qx)\Big|^2 (1+\log^2|x|)^{\delta/2}\,|x|\,d^*x \tag{1.24}$$

where the integral is performed on A^*/k^* and d^*x is the multiplicative Haar measure on A^*/k^*. The term $(1+\log^2|x|)^{\delta/2}$ is there to control the growth of the functions. The key point is that we use the measure $|x|\,d^*x$ instead of the additive Haar measure dx. Of course for a local field K one has $dx = |x|\,d^*x$ but this fails in the above global situation. Instead one has

$$dx = \lim_{\varepsilon\to 0}\ \varepsilon\,|x|^{1+\varepsilon}\,d^*x\,. \tag{1.25}$$

One has a natural representation of C_k on $L^2_\delta(X)_0$ given by

$$(U(j)\,f)\,(x) = f(j^{-1}\,x) \qquad \forall\,x\in A\ ,\ j\in C_k \tag{1.26}$$

and the result is independent of the choice of a lift of j in $J_k = \mathrm{GL}_1(A)$ because the functions $f - f_q$ are in the kernel of the norm. The conditions (1.23) which define $\mathcal{S}(A)_0$ are invariant under the action of C_k and give the following action of C_k on the 2-dimensional supplement of $\mathcal{S}(A)_0 \subset \mathcal{S}(A)$; this supplement is $\mathbb{C}\oplus\mathbb{C}(1)$ where $\mathbb{C}$ is the trivial C_k module (corresponding to $f(0)$) while the Tate twist $\mathbb{C}(1)$ is the module

$$(j,\lambda) \to |j|\,\lambda \tag{1.27}$$

coming from the equality

$$\int f(j^{-1}\,x)\,d\,x = |j| \int f(x)\,dx\,. \tag{1.28}$$

In order to analyze the representation (1.26) of C_k on $L^2_\delta(X)_0$ we shall relate it to the left regular representation of the group C_k on the Hilbert space $L^2_\delta(C_k)$ obtained from the following Hilbert space square norm on functions,

$$\|\xi\|^2_\delta = \int_{C_k} |\xi(g)|^2\,(1+\log^2|g|)^{\delta/2}\,d^*g \tag{1.29}$$

where we have normalized the Haar measure of the multiplicative group C_k, with module,

$$| \, | : C_k \to \mathbb{R}_+^* \tag{1.30}$$

in such a way that (cf. [24])

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty \, . \tag{1.31}$$

The left regular representation V of C_k on $L^2_\delta(C_k)$ is

$$(V(a)\,\xi)\,(g) = \xi(a^{-1}\,g) \qquad \forall\, g, a \in C_k \, . \tag{1.32}$$

Note that because of the weight $(1 + \log^2|x|)^{\delta/2}$, this representation is *not* unitary but it satisfies the growth estimate

$$\|V(g)\| = 0\,(\log|g|)^{\delta/2} \quad \text{when} \quad |g| \to \infty \tag{1.33}$$

which follows from the inequality (valid for $u, v \in \mathbb{R}$)

$$\rho(u+v) \leq 2^{\delta/2}\,\rho(u)\,\rho(v) \ , \ \rho(u) = (1+u^2)^{\delta/2} \, . \tag{1.34}$$

We let E be the linear isometry from $L^2_\delta(X)_0$ into $L^2_\delta(C_k)$ given by the equality,

$$E(f)\,(g) = |g|^{1/2} \sum_{q\in k^*} f(qg) \qquad \forall\, g \in C_k \, . \tag{1.35}$$

By comparing (1.24) with (1.29) we see that E is an isometry and the factor $|g|^{1/2}$ is dictated by comparing the measures $|g|\,d^*g$ of (1.24) with d^*g of (1.29).

One has $E(U(a)\,f)\,(g) = |g|^{1/2} \sum_{k^*} (U(a)\,f)\,(qg) = |g|^{1/2} \sum_{k^*} f(a^{-1}\,qg)$ $= |a|^{1/2}\,|a^{-1}\,g|^{1/2} \sum_{k^*} f(q\,a^{-1}\,g) = |a|^{1/2}\,(V(a)\,E(f))\,(g)$.

Thus,

$$E\,U(a) = |a|^{1/2}\,V(a)\,E \, . \tag{1.36}$$

This equivariance shows that the range of E in $L^2_\delta(C_k)$ is a closed invariant subspace for the representation V.

The following theorem and its corollary shows that the cokernel $\mathcal{H} = L^2_\delta(C_k)/\operatorname{Im}(E)$ of the isometry E plays the role of the Polya-Hilbert space. Since $\operatorname{Im} E$ is invariant under the representation V we let W be the corresponding representation of C_k on $\mathcal{H}$.

The Abelian locally compact group C_k is (non canonically) isomorphic to $K \times N$ where

$$K = \{g \in C_k \ ; \ |g| = 1\} \ , \ N = \text{range} \, | \, | \subset \mathbb{R}_+^* \, . \tag{1.37}$$

For number fields one has $N = \mathbb{R}_+^*$ while for fields of non zero characteristic $N \simeq \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}_+^*$.(Where $q = p^\ell$ is the cardinality of the field of constants).

We choose (non canonically) an isomorphism

$$C_k \simeq K \times N\,. \tag{1.38}$$

By construction the representation W satisfies (using (1.33)),

$$\|W(g)\| = 0(\log|g|)^{\delta/2} \tag{1.39}$$

and its restriction to K is unitary. Thus $\mathcal{H}$ splits as a canonical direct sum of pairwise orthogonal subspaces,

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{K}} \mathcal{H}_\chi \ , \ \mathcal{H}_\chi = \{\xi \ ; \ W(g)\,\xi = \chi\,(g)\,\xi \ , \ \forall\, g \in K\} \tag{1.40}$$

where χ runs through the Pontrjagin dual group of K, which is the discrete Abelian group $\widehat{K}$ of characters of K. Using the non canonical isomorphism (1.38), i.e. the corresponding inclusion $N \subset C_k$ one can now restrict the representation W to any of the sectors $\mathcal{H}_\chi$. When $\mathrm{char}(k) > 0$, then $N \simeq \mathbb{Z}$ and the condition (1.39) shows that the action of N on $\mathcal{H}_\chi$ is given by a single operator with *unitary* spectrum. (One uses the spectral radius formula $|\mathrm{Spec}\, w| = \overline{\mathrm{Lim}}\, \|\, w^n\|^{1/n}$.) When $\mathrm{Char}(k) = 0$, we are dealing with an action of $\mathbb{R}_+^* \simeq \mathbb{R}$ on $\mathcal{H}_\chi$ and the condition (1.39) shows that this representation is generated by a closed unbounded operator D with purely imaginary spectrum. The resolvent $R_\lambda = (D - \lambda)^{-1}$ is given, for $\mathrm{Re}\,\lambda > 0$, by the equality

$$R_\lambda = \int_0^\infty W_\chi(e^s)\, e^{-\lambda s}\, ds \tag{1.41}$$

and for $\mathrm{Re}\,\lambda < 0$ by,

$$R_\lambda = \int_0^\infty W_\chi(e^{-s})\, e^{\lambda s}\, ds \tag{1.42}$$

while the operator D is defined by

$$D\,\xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\,(W_\chi(e^\varepsilon) - 1)\,\xi\,. \tag{1.43}$$

Theorem 1. *Let $\chi \in \widehat{K}$, $\delta > 1$, $\mathcal{H}_\chi$ and D be as above. Then D has discrete spectrum, $\mathrm{Sp}\, D \subset i\,\mathbb{R}$ is the set of zeros of the L function with Grossencharacter $\widetilde{\chi}$ which have real part equal to $\frac{1}{2}$; $\rho \in \mathrm{Sp}\, D \Leftrightarrow L\left(\widetilde{\chi}, \frac{1}{2} + \rho\right) = 0$ and $\rho \in i\,\mathbb{R}$, where $\widetilde{\chi}$ is the unique extension of χ to C_k which is equal to 1 on N. Moreover the multiplicity of ρ in $\mathrm{Sp}\, D$ is equal to the largest integer $n < \frac{1+\delta}{2}$, $n \leq$ multiplicity of $\frac{1}{2} + \rho$ as a zero of L.*

Corollary 2. *For any Schwartz function $h \in \mathcal{S}(C_k)$ the operator $W(h) = \int W(g)\, h(g)\, d^*g$ in $\mathcal{H}$ is of trace class, and its trace is given by*

$$\text{Trace}\, W(h) = \sum_{\substack{L\left(\chi, \frac{1}{2}+\rho\right)=0 \\ \rho \in i\,\mathbb{R}}} \widehat{h}(\chi, \rho) \tag{1.44}$$

where the multiplicity is counted as in Theorem 1 and where the Fourier transform $\widehat{h}$ of h is defined by,

$$\widehat{h}(\chi, \rho) = \int_{C_k} h(u)\, \tilde{\chi}(u)\, |u|^{\rho}\, d^*u\,. \tag{1.45}$$

Note that we did not have to define the L functions before stating the theorem, which shows that the pair

$$(\mathcal{H}_\chi, D) \tag{1.46}$$

certainly qualifies as a Polya-Hilbert space.

The case of the Riemann zeta function corresponds to the trivial character $\chi = 1$ for the global field $k = \mathbb{Q}$ of rational numbers.

In general the zeros of the L functions can have multiplicity but one expects that for a fixed Grossencharacter χ this multiplicity is bounded, so that for a large enough value of δ the spectral multiplicity of D will be the right one. When the characteristic of k is > 0 this is certainly true.

If we modify the choice of non canonical isomorphism (1.38) this modifies the operator D by

$$D' = D - i\,s \tag{1.47}$$

where $s \in \mathbb{R}$ is determined by the equality

$$\tilde{\chi}'(g) = \tilde{\chi}(g)\, |g|^{i\,s} \qquad \forall\, g \in C_k\,. \tag{1.48}$$

The coherence of the statement of the theorem is insured by

$$L(\tilde{\chi}', z) = L(\tilde{\chi}, z + i\,s) \qquad \forall\, z \in \mathbb{C}\,. \tag{1.49}$$

When the zeros of L have multiplicity and δ is large enough the operator D is *not* semisimple and has a non trivial Jordan form. This is compatible with the almost unitary condition (1.39) but not with skew symmetry for D.

The proof of theorem 1 [7] is based on the distribution theoretic interpretation by A. Weil [23] of the idea of Tate and Iwasawa on the functional equation. Our construction should be compared with [3] and [25].

As we expected from (1.16), the Polya-Hilbert space $\mathcal{H}$ appears as a cokernel. Since we obtain the Hilbert space $L^2_\delta(X)_0$ by imposing two linear conditions on $\mathcal{S}(A)$,

$$0 \to \mathcal{S}(A)_0 \to \mathcal{S}(A) \xrightarrow{L} \mathbb{C} \oplus \mathbb{C}(1) \to 0 \tag{1.50}$$

we shall define $L^2_\delta(X)$ so that it fits in an exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0\,. \tag{1.51}$$

We can then use the exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(C_k) \to \mathcal{H} \to 0 \tag{1.52}$$

together with Corollary 2 to compute in a formal manner what the character of the module $L^2_\delta(X)$ should be. Using (1.51) and (1.52) we obtain,

$$\text{"Trace"}\ (U(h)) = \widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\rho=\frac{1}{2}}} \widehat{h}(\chi,\rho) + \infty\, h(1) \tag{1.53}$$

where $\widehat{h}(\chi,\rho)$ is defined by Corollary 2 and

$$U(h) = \int_{C_k} U(g)\, h(g)\, d^* g \tag{1.54}$$

while the test function h is in a suitable function space. Note that the trace on the left hand side of (1.53) only makes sense after a suitable regularization since the left regular representation of C_k is not traceable. This situation is similar to the one encountered by Atiyah and Bott [1] in their proof of the Lefchetz formula. In particular it is important to deal not with Hilbert spaces but rather with nuclear spaces in the sense of Grothendieck. The point being that the Schwartz kernel theorem is then available and one can at least talk about the integral of the diagonal values of the Schwartz kernels as a problem of product of distributions. In our context this is achieved by letting δ go to ∞, i.e. by considering

$$\mathcal{S}(X) = \bigcap_{\delta} L^2_\delta(X)\,. \tag{1.55}$$

This space is *locally* nuclear for the action of C_k. In particular the Schwartz kernel theorem applies to the operators $U(h)$.

2 The distribution trace formula for flows on manifolds

In order to understand how the left hand side of (1.53) should be computed we shall first give a leisurely account of the much easier but analogous computation of the distribution theoretic trace for flows on manifolds,which is a variation on the theme of [1]. We just follow Guillemin Sternberg [10] and extract from [10] the relevant case for our discussion.

Recall that given a vector space E over $\mathbb{R}$, $\dim E = n$, a density is a map, $\rho \in |E|$,

$$\rho : \wedge^n E \to \mathbb{C} \tag{2.1}$$

such that $\rho(\lambda v) = |\lambda|\, \rho(v) \qquad \forall\, \lambda \in \mathbb{R}$.

Given a linear map $T : E \to F$ we let $|T| : |F| \to |E|$ be the corresponding linear map,it depends contravariantly on T.

Given a manifold M and $\rho \in C_c^\infty(M, |TM|)$ one has a canonical integral,

$$\int \rho \in \mathbb{C}\,. \tag{2.2}$$

Given a vector bundle L on M one defines the generalized sections on M as the dual space of $C_c^\infty(M, L^* \otimes |TM|)$

$$C^{-\infty}(M, L) = \text{ dual of } C_c^\infty(M, L^* \otimes |TM|) \tag{2.3}$$

where L^* is the dual bundle. One has a natural inclusion,

$$C^\infty(M, L) \subset C^{-\infty}(M, L) \tag{2.4}$$

given by the pairing

$$\sigma \in C^\infty(M, L)\ ,\ s \in C_c^\infty(M, L^* \otimes |TM|) \to \int \langle s, \sigma\rangle \tag{2.5}$$

where $\langle s, \sigma\rangle$ is viewed as a density, $\langle s, \sigma\rangle \in C_c^\infty(M, |TM|)$.

One has a similar notion of generalized section with compact support.

Given a smooth map $\varphi : X \to Y$, then if φ is *proper*, it gives a (contravariantly) associated map

$$\varphi^* : C_c^\infty(Y, L) \to C_c^\infty(X, \varphi^*(L))\ ,\ (\varphi^*\, \xi)(x) = \xi(\varphi(x)) \tag{2.6}$$

where $\varphi^*(L)$ is the pull back of the vector bundle L.

Thus, given a linear form on $C_c^\infty(X, \varphi^*(L))$ one has a (covariantly) associated linear form on $C_c^\infty(Y, L)$. In particular with L trivial we see that given a generalized density $\rho \in C^{-\infty}(X, |T|)$ one has a push forward

$$\varphi_*(\rho) \in C^{-\infty}(Y, |T|) \tag{2.7}$$

with $\langle \varphi_*(\rho), \xi \rangle = \langle \rho, \varphi^* \xi \rangle \qquad \forall \xi \in C_c^\infty(X)$.

Next, if φ is a fibration and $\rho \in C_c^\infty(X, |T|)$ is a density then one can integrate ρ along the fibers, the obtained density on Y, $\varphi_*(\rho)$ is given as in (2.7) by

$$\langle \varphi_*(\rho), f \rangle = \langle \rho, \varphi^* f \rangle \qquad \forall f \in C^\infty(Y) \tag{2.8}$$

but the point is that it is not only a generalized section but a smooth section $\varphi_*(\rho) \in C_c^\infty(Y, |T|)$.

It follows that if $f \in C^{-\infty}(Y)$ is a generalized function, then one obtains a generalized function $\varphi^*(f)$ on X by,

$$\langle \varphi^*(f), \rho \rangle = \langle f, \varphi_*(\rho) \rangle \qquad \forall \rho \in C_c^\infty(X, |T|)\,. \tag{2.9}$$

In general,the pullback $\varphi^*(f)$ continues to make sense provided the following transversality condition holds,

$$d(\varphi^*(l)) \neq 0 \quad \forall l \in WF(f)\,. \tag{2.10}$$

where $WF(f)$ is the wave front set of f [10]. The next point is the construction of the generalized section of a vector bundle L on a manifold X associated to a submanifold $Z \subset X$ and a symbol,

$$\sigma \in C^\infty(Z, L \otimes |N_Z^*|)\,. \tag{2.11}$$

where N_Z is the normal bundle of Z. The construction is the same as that of the current of integration on a cycle. Given $\xi \in C_c^\infty(X, L^* \otimes |T|)$, the product $\sigma\,\xi/Z$ is a density on Z, since it is a section of $|T_Z| = |T_X| \otimes |N_Z^*|$.One can thus integrate it over Z. When $Z = X$ one has $N_Z^* = \{0\}$ and $|N_Z^*|$ has a canonical section, so that the current associated to σ is just given by (2.5). When $Z = \text{pt}$ is a single point $x \in X$ a generalized section of L given by a Dirac distribution at x requires not only a vector $\xi_x \in L_x$ but also a dual density, i.e. a volume multivector $v \in |T_x^*|$.

Now let $\varphi : X \to Y$ with Z a submanifold of Y and σ as in (2.11).

Let us assume that φ is transverse to Z, so that for each $x \in X$ with $y = \varphi(x) \in Z$ one has

$$\varphi_*(T_x) + T_{\varphi(x)}(Z) = T_y\, Y\,. \tag{2.12}$$

Let

$$\tau_x = \{X \in T_x\ ,\ \varphi_*(X) \in T_y(Z)\}\,. \tag{2.13}$$

Then φ_* gives a canonical isomorphism,

$$\varphi_* : T_x(X)/\tau_x \simeq T_y(Y)/T_y(Z) = N_y(Z)\,. \tag{2.14}$$