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Mapping Degree Theory

To Jesús María Ruiz Amestoy

Preface

This book springs from lectures on degree theory given by the authors during many years at the Departamento de Geometría y Topología at the Universidad Complutense de Madrid, and its definitive form corresponds to a three-month course given at the Dipartimento di Matematica at the Università di Pisa during the spring of 2006. Today mapping degree is a somewhat classical topic that appeals to geometers and topologists for its beauty and ample range of relevant applications. Our purpose here is to present both the history and the mathematics.

The notion of degree was discovered by the great mathematicians of the decades around 1900: Cauchy, Poincaré, Hadamard, Brouwer, Hopf, etc. It was then brought to maturity in the 1930s by Hopf and also by Leray and Schauder. The theory was fully burnished between 1950 and 1970. This process is described in Chapter I. As a complement, at the end of the book there is included an index of names of the mathematicians who played their part in the development of mapping degree theory, many of whom stand tallest in the history of mathematics. After the first historical chapter, Chapters II, III, IV, and V are devoted to a more formal proposition-proof discourse to define and study the concept of degree and its applications. Chapter II gives a quick presentation of manifolds, with special emphasis on aspects relevant to degree theory, namely regular values of differentiable mappings, tubular neighborhoods, approximation, and orientation. Although this chapter is primarily intended to provide a review for the reader, it includes some not so standard details, for instance concerning tubular neighborhoods. The main topic, degree theory, is presented in Chapters III and IV. In a simplified manner we can distinguish two approaches to the theory: the Brouwer-Kronecker degree and the Euclidean degree. The first is developed in Chapter III by differential means, with a quick diversion into the de Rham computation in cohomological terms. We cannot help this diversion: cohomology is too appealing to skip. Among other applications, we obtain in this chapter a differential version of the Jordan Separation Theorem. Then, we construct the Euclidean degree in

Chapter IV. This is mainly analytic and astonishingly simple, especially in view of its extraordinary power. We hope this partisan claim will be acknowledged readily, once we obtain quite freely two very deep theorems: the Invariance of Domain Theorem and the Jordan Separation Theorem, the latter in its utmost topological generality. Finally, Chapter V is devoted to some of those special results in mathematics that justify a theory by their depth and perfection: the Hopf and the Poincaré-Hopf Theorems, with their accompaniment of consequences and comments. We state and prove these theorems, which gives us the perfect occasion to take a glance at tangent vector fields.

We have included an assorted collection of some 180 problems and exercises distributed among the sections of Chapters II to V, none for Chapter I due to its nature. Those problems and exercises, of various difficulty, fall into three different classes: (i) suitable examples that help to seize the ideas behind the theory, (ii) complements to that theory, such as variations for different settings, additional applications, or unexpected connections with different topics, and (iii) guides for the reader to produce complete proofs of the classical results presented in Chapter I, once the proper machinery is developed.

We have tried to make internal cross-references clearer by adding the Roman chapter number to the reference, either the current chapter number or that of a different chapter. For example, III.6.4 refers to Proposition 6.4 in Chapter III; similarly, the reference IV.2 means Section 2 in Chapter IV. We have also added the page number of the reference in most cases.

One essential goal of ours must be noted here: we attempt the simplest possible presentation at the lowest technical cost. This means we restrict ourselves to elementary methods, whatever meaning is accepted for elementary. More explicitly, we only assume the reader is acquainted with basic ideas of differential topology, such as can be found in any text on the calculus on manifolds.

We only hope that this book succeeds in presenting degree theory as it deserves to be presented: we view the theory as a genuine masterpiece, joining brilliant invention with deep understanding, all in the most accomplished attire of clarity. We have tried to share that view of ours with the reader.

Los Molinos and Majadahonda
June 2009

E. Outerelo and J.M. Ruiz

Contents

History

In the body of mathematics, the notion of degree stands as a beautiful achievement of topology and one of the main contributions of the twentieth century, which has been called the century of topology. In Chapter I we try to outline how the ideas that led to this fundamental notion of degree were sparked and came to light. It is only natural that such a task is biased by our personal opinions and preferences. Thus, it is likely that a specialist in, say, partial differential equations would present the tale in a somewhat different way. All in all, a choice must be made and ours is this:

§1. *Prehistory:* Gauss, Cauchy, Liouville, Sturm, Kronecker, Poincaré, Picard, Bohl (1799–1910).

§2. *Inception and formation:* Hadamard, Brouwer (1910–1912).

§3. *Accomplishment:* Hopf, Leray, Schauder (1925–1934).

§4. *Renaissance and reformation:* Nagumo, de Rham, Heinz (1950–1970).

§5. *Axiomatization:* Führer, Amann, Weiss (1970–1972).

§6. *Further developments:* Equivariant theory, infinite dimensions.

The presentation of these topics is mainly discursive and descriptive, rigorous proofs being deferred to Chapters II through V where there will be complete arguments for all the most classical results presented here.

1. Prehistory

Rougly speaking, degree theory can be defined as the study of those techniques that give information *on the existence of solutions of an equation of the form* $y = f(x)$, where x and y dwell in suitable spaces and f is a map from one to the other. The theory also gives clues for *the number of solutions and their nature.* An important particular case is that of an equation $x = f(x)$, where f is a map from a domain D of a linear space into D itself: this is the so-called *Fixed Point Problem.*

By its very nature, it is clear that the origins of degree theory should be traced back to the first attempts to solve algebraic equations such as

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0,$$

where the coefficients a_i are complex numbers, $a_n \neq 0$. That such an equation always has some solution is the *Fundamental Theorem of Algebra.* This result was most beloved by KARL-FRIEDRICH GAUSS (1777–1855), who found at least four different proofs, in 1799, 1815, 1816, and 1849. It is precisely in the first and fourth proofs where we find what can be properly considered the first ideas of topological degree. By some properties of algebraic curves (which were formalized only in 1933 by ALEXANDER OSTROWSKI (1893–1986)), Gauss was able to prove that inside a circle of big enough radius, the algebraic curve corresponding to the real part of the polynomial shares some point with the algebraic curve corresponding to the imaginary part. In this way the following two lines of research were born:

Problem I. Find the common solutions of the equations

$$\begin{cases} f(x,y) = 0, \\ F(x,y) = 0 \end{cases}$$

inside a given closed planar domain, on whose border the two functions $f(x,y)$ and $F(x,y)$ do not vanish simultaneously.

Problem II. Find the number of real roots of a polynomial in one variable with real coefficients, in a given closed interval $[a,b]$ of the real line.

* * *

The first contributions to **Problem I** are due to AUGUSTIN LOUIS CAUCHY (1789–1857). In a memoir presented before the Academy of Turin, on November 17, 1831, and in the paper [Cauchy 1837a], Cauchy introduces a new calculus that, in its own words, can be used to solve equations of any degree.

Some parts of Cauchy's arguments are not completely precise, and the way these parts were made rigorous by JACQUES CHARLES FRANÇOIS STURM (1803–1855) and JOSEPH LIOUVILLE (1809–1882) is quite relevant in the history of the analytic definition of the topological degree of a continuous mapping.

Let us describe this. The definition of the *index of a function* given by Cauchy in [Cauchy 1837a] is the following:

Let x be a real variable and $f(x)$ a function that becomes infinite at $x = a$. If the variable x increases through a, the function will either change from negative to positive or change from positive to negative or not change sign at all. We will say that the index of f at a *is* -1 *in the first case,* $+1$ *in the second, and* 0 *in the third. We define the* integral index of f *between two given limits $x = x_0$ and $x = x_1$, denoted by $J_{x_0}^{x_1}(f)$, as the sum of the indices of f corresponding to the values of x in the interval $[x_0, x_1]$ at which f becomes infinite. If f is a function in two variables, we define the* integral index of f *between the limits $x_0, x_1; y_0, y_1$ to be the number*

$$ {}_{x_0}^{x_1}J_{y_0}^{y_1}(f) = \tfrac{1}{2}\left[J_{x_0}^{x_1}(f(\cdot, y_1)) - J_{x_0}^{x_1}(f(\cdot, y_0)) - J_{y_0}^{y_1}(f(x_1, \cdot)) + J_{y_0}^{y_1}(f(x_0, \cdot))\right]. $$

In his 1831 memoir, Cauchy obtained the index of a function by integral techniques and residues and proved the following result:

Theorem. *Let Γ be a closed curve that is the contour of an area S, and let $Z(z) = X(x, y) + iY(x, y)$ be an entire function. Then*

$$ \frac{1}{2\pi i}\int_\Gamma \frac{Z'(z)}{Z(z)}dz = \tfrac{1}{2}\, J_{s=s'}^{s=s''}\left(X/Y\right) $$

is the number of zeros of $Z(z)$ in S; here s stands for the arc length along Γ, and $s'' - s'$ is the length of Γ.

Cauchy generalized this result in a memoir published June 16, 1833, in Turin. The generalization follows:

Theorem. *Let $F(x, y)$ and $f(x, y)$ be two functions of the variables x, y, continuous between the limits $x = x_0, x = x_1, y = y_0, y = y_1$. We denote by $\Phi(x, y), \phi(x, y)$ the derivatives of the functions with respect to x, and by $\Psi(x, y), \psi(x, y)$ their derivatives with respect to y. Finally, let N be the number of the different systems of values x, y, between the above limits, verifying simultaneously the equations $F(x, y) = 0, f(x, y) = 0$. Then*

$$ N = {}_{x_0}^{x_1}J_{y_0}^{y_1}(\Delta), $$

where

$$ \begin{aligned} \Delta(x, y) &= \frac{f(x, y)}{F(x, y)}\left(\Phi(x, y)\psi(x, y) - \Psi(x, y)\phi(x, y)\right) \\ &= \frac{f(x, y)}{F(x, y)}\left(\frac{\partial F(x, y)}{\partial x}\frac{\partial f(x, y)}{\partial y} - \frac{\partial F(x, y)}{\partial y}\frac{\partial f(x, y)}{\partial x}\right). \end{aligned} $$

An elementary "proof" of this theorem appears in [Cauchy 1837b]. However, Liouville and Sturm in [Liouville-Sturm 1837] give three examples showing that the second theorem above above can fail. The first example is

$$\begin{cases} F(x,y) = x^2 + y^2 - 1, \\ f(x,y) = y. \end{cases}$$

In this example

$$\Delta(x,y) = \frac{2xy}{x^2 + y^2 - 1},$$

and drawing around the origin a rectangle containing the circle $x^2 + y^2 = 1$, one sees that

$${}_{x_0}^{x_1}J_{y_0}^{y_1}(\Delta) = 0,$$

because Δ never becomes infinity on the sides of the rectangle. However, the system

$$\begin{cases} x^2 + y^2 - 1 = 0, \\ y = 0 \end{cases}$$

has the two solutions $(1,0), (-1,0)$ inside the rectangle. Liouville and Sturm conclude their note with the following remark:

There is a theorem that can replace Cauchy's. Let us consider a closed contour Γ on which $F(x,y)$ and $f(x,y)$ do not vanish simultaneously, and let us also assume that inside this contour the function

$$\begin{aligned} w &= \Phi(x,y)\psi(x,y) - \Psi(x,y)\phi(x,y) \\ &= \frac{\partial F(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y} - \frac{\partial F(x,y)}{\partial y}\frac{\partial f(x,y)}{\partial x} \end{aligned}$$

does not vanish at the values (x,y) at which $f(x,y)$ and $F(x,y)$ vanish.

In this situation, among the solutions (x,y) of the equations $F(x,y) = 0$, $f(x,y) = 0$, inside Γ, some correspond to positive values of w and others to negative values of w. We denote by μ_1 the number of solutions of the first kind, and by μ_2 the number of solutions of the second kind. With this notation we have

$$\frac{1}{2}\delta = \mu_1 - \mu_2,$$

where δ stands for how many more times the function $\frac{f(x,y)}{F(x,y)}$ changes from positive to negative than from negative to positive, at those points in the contour Γ at which that function becomes infinite, when the contour is traced in the positive direction.

We see that the function w is the *Jacobian of the mapping* (F, f) (Liouville and Sturm always consider entire functions). Consequently, we find displayed here for the first time the importance of the sign of the functional determinant

$$w = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{vmatrix}$$

when dealing with the computation of the number of solutions of the system

$$\begin{cases} F(x,y) = 0, \\ f(x,y) = 0 \end{cases}$$

in a planar region.

Today, in the hypotheses of the Liouville-Sturm Theorem, the number $\mu_1 - \mu_2$ is called the *topological degree of the mapping* (F, f) *at the origin,* and this is the starting point for the analytic definition of degree. But this will not take full shape until 1951.

* * *

In the later paper [Cauchy 1855], Cauchy states the *Argument Principle,* which is another way to compute the indices he has defined earlier. These results, translated into more modern terminology, read as follows.

Winding number (or index) of a planar curve around a point. *Let* $\Gamma \subset \mathbb{C}$ *be a closed oriented curve with a* C^1 *parametrization:*

$$z(t) = x(t) + iy(t) + a, \quad 0 \le t \le 1, \quad z(0) = z(1), \quad a \in \mathbb{C} \setminus \Gamma.$$

Then,

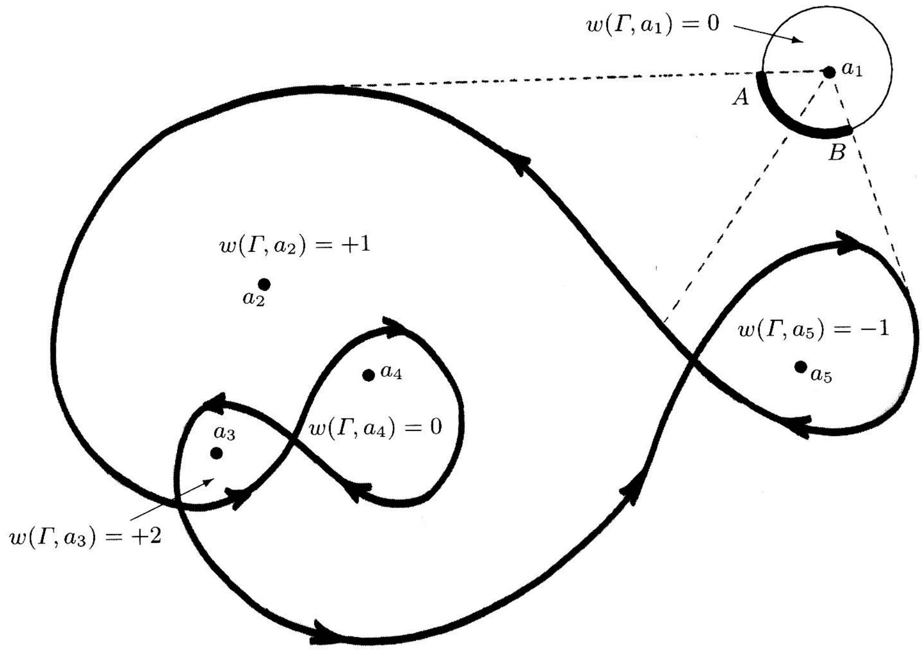
$$w(\Gamma, a) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-a} = \frac{1}{2\pi}\int_0^1 \frac{x(t)y'(t) - x'(t)y(t)}{x^2(t) + y^2(t)}dt$$

is an integer.

This integer is called the *winding number (or index) of* Γ *around* a.

Geometrically, the winding number tells us *how many times the curve wraps around the point.* In case Γ is only continuous, the winding number is defined through a C^1 approximation Γ_1 of Γ, because $w(\Gamma_1, a)$ remains constant for Γ_1 close enough to Γ.

The following example illustrates this notion:



To proceed one step further, Cauchy considers a simply connected domain $G \subset \mathbb{C}$ (that is, G has no holes), a holomorphic function $f : G \to \mathbb{C}$, $\zeta = f(z)$, and a $\mathcal{C}^1$ closed curve $\Gamma \subset G$, on which f has no zeros. Then:

Argument Principle. *The following formula holds:*

$$w(f(\Gamma), 0) = \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}dz = \sum_k w(\Gamma, a_k)\alpha_k,$$

where the a_k's are the zeros of f in the domain D bounded by Γ and the α_k's are their respective multiplicities.

Suppose next that Γ has no self-intersection and that it has the positive (counterclockwise) orientation. Then D is a connected domain (this is the *Jordan Separation Theorem,* which we will discuss later), and $w(\Gamma, a) = +1$ for all $a \in D$, so that the last formula becomes

$$w(f(\Gamma), 0) = \sum_k \alpha_k,$$

that is:

Theorem. *The total number of zeros (counted with multiplicities) that f has in D is the winding number of the curve $f(\Gamma)$ around the origin.*

In general, the winding number can be negative, but we can still say that f has at least $|w(f(\Gamma), 0)|$ zeros in the domain bounded by Γ.

* * *

Let us now turn to **Problem II.** The first full solution is due to Sturm. In 1829 and 1835 he gave an algorithm to find the exact number of distinct real roots of a polynomial. The theorem was later generalized by CARL GUSTAV JACOB JACOBI (1804–1851), CHARLES HERMITE (1822–1901), and JAMES JOSEPH SYLVESTER (1814–1897).

Exploring the topological content of Sylvester's article [Sylvester 1853], LEOPOLD KRONECKER (1823–1891) introduces in his papers [Kronecker 1869a] and [Kronecker 1869b] a method that extends Sturm's. Indeed, at the end of his work Kronecker writes:

In my research developed in this article, I started from a theorem by Sturm. A generalization of that result was found by Hermite some time ago, but I have been able to extend the continued fraction algorithm developed by Sylvester to further widen Sturm's theorem.

Let us describe Kronecker's contribution. He starts with the following definition:

Regular function systems. A *regular function system* consists of $n+1$ real functions $F_0, F_1, \ldots, F_n$ in n real variables $x_1, \ldots, x_n$, such that

(a) $F_0, F_1, \ldots, F_n$ are continuous and have no common zeros. They admit partial derivatives with respect to all n variables, and those derivatives take finite values.

(b) The functions $F_0, F_1, \ldots, F_n$ take positive and negative values. Moreover, each function takes positive (resp., negative) values infinitely often.

(c) The domains $\{F_i < 0\}$, $i = 0, \ldots, n$, represent n-dimensional varieties that only contain finite values of the variables $x_1, \ldots, x_n$.

(d) No functional determinant

$$\left|\frac{\partial F_i}{\partial x_j}\right|_{\substack{k \neq i = 0,1,\ldots,n \\ j=1,\ldots,n}}, \qquad k = 0, 1, \ldots, n,$$

vanishes at any zero of the system $F_k \neq 0, F_0 = F_1 = \cdots = F_n = 0$.

(e) The common zero set of any chosen $n-1$ functions among $F_0, F_1, \ldots, F_n$ is a $\mathcal{C}^1$ curve.

Then Kronecker looks at the orientations of the $\mathcal{C}^1$ curve involved in this definition (condition (e) above). He considers this part basic in his research on systems of functions in several variables:

Orientation Principle. Kronecker chooses for every pair (k,ℓ), $k<\ell$, an orientation of the $\mathcal{C}^1$ curve (recall (e) above)

$$F(k,\ell)=\{x\in\mathbb{R}^n:\ F_i(x)=0\ \ \text{for } i\neq k,\ell\}.$$

This orientation is denoted by $|k\ell|$; he then puts $|\ell k|=-|k\ell|$.

Next, he defines:

(a) A point $e\in F(k,\ell)\cap\{F_k=0\}$ is called an *incoming (eingang)* point of $F(k,\ell)$ (in $\{x\in\mathbb{R}^n: F_k(x)\cdot F_\ell(x)<0\}$) if the following condition holds true: *walking the curve $F(k,\ell)$ as oriented by $|k\ell|$, we leave the set $\{x\in\mathbb{R}^n : F_k(x)\cdot F_\ell(x)>0\}$ at the point e and enter $\{x\in\mathbb{R}^n : F_k(x)\cdot F_\ell(x)<0\}$.*

The set of all these incoming points e is denoted by $E(k,\ell)$.

(b) A point $a\in F(k,\ell)\cap\{F_k=0\}$ is called an *outgoing (ausgang)* point of $F(k,\ell)$ (off $\{x\in\mathbb{R}^n:\ F_k(x)\cdot F_\ell(x)<0\}$) if the following condition holds true: *walking the curve $F(k,\ell)$ as oriented by $|k\ell|$, we leave the set $\{x\in\mathbb{R}^n :\ F_k(x)\cdot F_\ell(x)<0\}$ at the point a and enter $\{x\in\mathbb{R}^n : F_k(x)\cdot F_\ell(x)>0\}$.*

The set of all these outgoing points a is denoted by $A(k,\ell)$.

After the preceding preparation, Kronecker shows that the number

$$\#E(k,\ell)-\#A(k,\ell)$$

is *even* and does not depend on the indices k,ℓ, and then he defines:

Kronecker characteristic. The *characteristic of the regular function system $F_0, F_1, \ldots, F_n$* is the integer

$$\chi(F_0,F_1,\ldots,F_n)=\tfrac{1}{2}\big(\#E(k,\ell)-\#A(k,\ell)\big).$$

It is convenient to stress that in the course of his proof of this fact