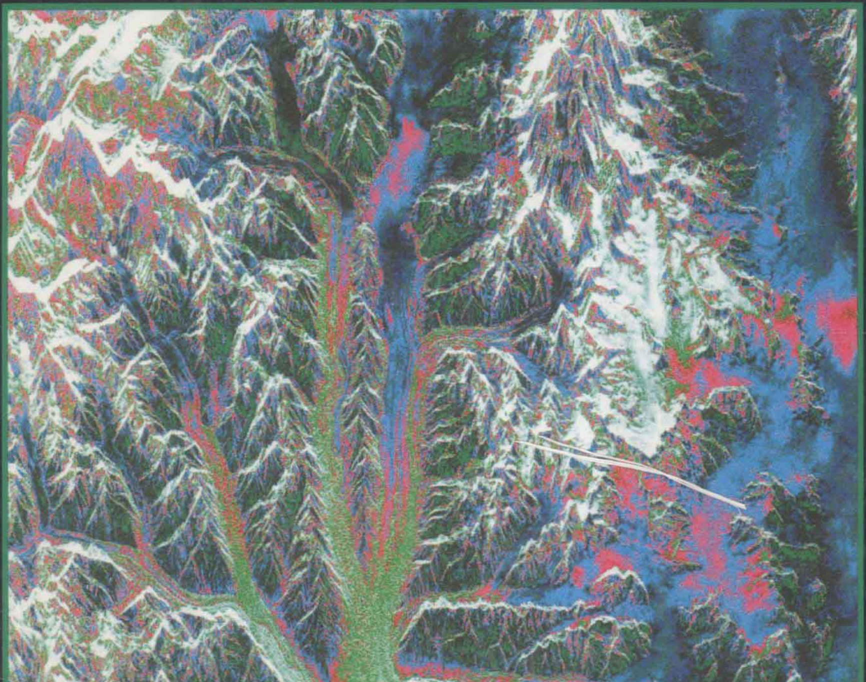
Jean-François Gouyet

Physics and Fractal Structures



Physics and Fractal Structures

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Foreword by Benoît MANDELBROT

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Foreword

When intellectual and political movements ponder their roots, no event looms larger than the first congress. The first meeting on fractals was held in July 1982 in Courchevel, in the French Alps, through the initiative of Herbert Budd and with the support of IBM Europe Institute. Jean-François Gouyet's book reminds me of Courchevel, because it was there that I made the acquaintance and sealed the friendship of one of the participants, Bernard Sapoval, and it was from there that the fractal bug was taken to École Polytechnique. Sapoval, Gouyet and Michel Rosso soon undertook the work that made their laboratory an internationally recognized center for fractal research. If I am recounting all this, it is to underline that Gouyet is not merely the author of a new textbook, but an active player on a world-famous stage. While the tone is straightforward, as befits a textbook, he speaks with authority and deserves to be heard.

The topic of fractal diffusion fronts which brought great renown to Gouyet and his colleagues at Polytechnique is hard to classify, so numerous and varied are the fields to which it applies. I find this feature to be particularly attractive. The discovery of fractal diffusion fronts can indeed be said to concern the theory of welding, where it found its original motivation. But it can also be said to concern the physics of (poorly) condensed matter. Finally it also concerns one of the most fundamental concepts of mathematics, namely, diffusion. Ever since the time of Fourier and then of Bachelier (1900) and Wiener (1922), the study of diffusion keeps moving forward, yet entirely new questions come about rarely. Diffusion fronts brought in something entirely new.

Returning to the book itself, if the variety of the topics comes as a surprise to the reader, and if the brevity of some of treatments leaves him or her hungry for more, then the author will have achieved the goal he set himself. The most important specialized texts treating the subject are carefully referenced and should satisfy most needs.

To sum up, I congratulate Jean-François warmly and wish his book the great success it deserves.

Benoît B. MANDELBROT

Yale University
IBM T.J. Watson Research Center

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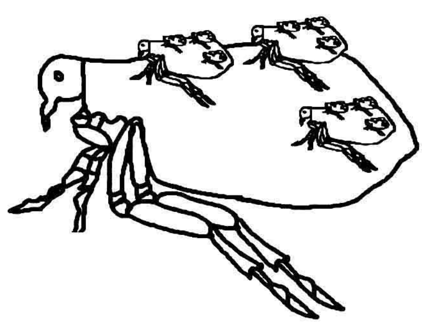
So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey,
And these have smaller yet to bite 'em
And so proceed ad infinitum.

...

Jonathan Swift, 1733,
On poetry, a Rhapsody.



Preface

The introduction of the concept of fractals by Benoît B. Mandelbrot at the beginning of the 1970's represented a major revolution in various areas of physics. The problems posed by phenomena involving fractal structures may be very difficult, but the formulation and geometric understanding of these objects has been simplified considerably. This no doubt explains the immense success of this concept in dealing with all phenomena in which a semblance of disorder appears.

Fractal structures were discovered by mathematicians over a century ago and have been used as subtle examples of continuous but *nonrectifiable* curves, that is, those whose length cannot be measured, or of continuous but *nowhere differentiable* curves, that is, those for which it is impossible to draw a tangent at any their points. Benoît Mandelbrot was the first to realize that many shapes in nature exhibit a fractal structure, from clouds, trees, mountains, certain plants, rivers and coastlines to the distribution of the craters on the moon. The existence of such structures in nature stems from the presence of disorder, or results from a functional optimization. Indeed, this is how trees and lungs maximise their surface/volume ratios.

This volume, which derives from a course given for the last three years at the Ecole Supérieure d'Electricité, should be seen as an introduction to the numerous phenomena giving rise to fractal structures. It is intended for students and for all those wishing to initiate themselves into this fascinating field where apparently disordered forms become geometry. It should also be useful to researchers, physicists, and chemists, who are not yet experts in this field.

This book does not claim to be an exhaustive study of all the latest research in the field, yet it does contains all the material necessary to allow the reader to tackle it. Deeper studies may be found not only in Mandelbrot's books (Springer Verlag will publish a selection of books which bring together reprints of published articles along with many unpublished papers), but also in the very abundant, specialized existing literature, the principal references of which are located at the end of this book.

The initial chapter introduces the principal mathematical concepts needed to characterize fractal structures. The next two chapters are given over to fractal geometries found in nature; the division of these two chapters is intended to help the presentation. Chapter 2 concerns those structures which may extend to enormous sizes (galaxies, mountainous reliefs, etc.), while Chap. 3 explains those fractal structures studied by materials physicists. This classification is obviously too rigid; for example, fractures generate similar structures ranging in size from several microns to several hundreds of meters.

In these two chapters devoted to fractal geometries produced by the physical world, we have introduced some very general models. Thus fractional Brownian motion is introduced to deal with reliefs, and percolation to deal with disordered media. This approach, which may seem slightly unorthodox seeing that these concepts have a much wider range of application than the examples to which they are attached, is intended to lighten the mathematical part of the subject by integrating it into a physical context.

Chapter 4 concerns growth models. These display too great a diversity and richness to be dispersed in the course of the treatment of the various phenomena described.

Finally, Chap. 5 introduces the dynamic aspects of transport in fractal media. Thus it completes the geometric aspects of dynamic phenomena described in the previous chapters.

I would like to thank my colleagues Pierre Collet, Eric Courtens, François Devreux, Marie Farge, Max Kolb, Roland Lenormand, Jean-Marc Luck, Laurent Malier, Jacques Peyrière, Bernard Sapoval, and Richard Schaeffer, for the many discussions which we have had during the writing of this book. I thank Benoît Mandelbrot for the many improvements he has suggested throughout this book and for agreeing to write the preface. I am especially grateful to Etienne Guyon, Jean-Pierre Hulin, Pierre Moussa, and Michel Rosso for all the remarks and suggestions that they have made to me and for the time they have spent in checking my manuscript. Finally, I would like to thank Marc Donnart and Suzanne Gouyet for their invaluable assistance during the preparation of the final version.

The success of the French original version published by Masson, has motivated Masson and Springer to publish the present English translation. I am greatly indebted to them. I acknowledge Dr. David Corfield who carried out this translation and Dr. Clarissa Javanaud and Prof. Eugene Stanley for many valuable remarks upon the final translation. During the last four years, the use of fractals has widely spread in various fields of science and technology, and some new approaches (such as wavelets transform) or concepts (such as scale relativity) have appeared. But the essential of fractal knowledge was already present at the end of the 1980s.

Palaiseau, July 1995

Contents

3. Natural fractal structures:... to the microscopic

4. Growth models

5. Dynamical aspects

CHAPTER 1

Fractal Geometries

1.1 Introduction

The end of the 1970s saw the idea of *fractal geometry* spread into numerous areas of physics. Indeed, the concept of fractal geometry, introduced by B. Mandelbrot, provides a solid framework for the analysis of natural phenomena in various scientific domains. As Roger Pynn wrote in *Nature*, "If this opinion continues to spread, we won't have to wait long before the study of fractals becomes an obligatory part of the university curriculum."

The *fractal* concept brings many earlier mathematical studies within a single framework. The objects concerned were invented at the end of the 19th century by such mathematicians as Cantor, Peano, etc. The term "*fractal*" was introduced by B. Mandelbrot (fractal, i.e., that which has been infinitely divided, from the Latin "*fractus*," derived from the verb "*frangere,*" to break). It is difficult to give a precise yet general definition of a fractal object; we shall define it, following Mandelbrot, as a set which shows irregularities on all scales.

Fundamentally it is its *geometric* character which gives it such great scope; fractal geometry forms the missing complement to Euclidean geometry and crystalline symmetry.[1] As Mandelbrot has remarked, clouds are not spheres, nor mountains cones, nor islands circles and their description requires a different geometrization.

As we shall show, the idea of fractal geometry is closely linked to properties invariant under change of scale: a fractal structure is the same "*from near or from far.*" The concepts of self-similarity and scale invariance appeared independently in several fields; among these, in particular, are critical phenomena and second order phase transitions.[2] We also find fractal geometries in particle trajectories, hydrodynamic lines of flux, waves, landscapes, mountains, islands and rivers, rocks, metals, and composite materials, plants, polymers, and gels, etc.

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[1] We must, however, add here the recent discoveries about quasicrystalline symmetries.

[2] We shall not refer here to the wide and fundamental literature on critical phenomena, renormalization, etc.

Many works on the subject have been published in the last 10 years. Basic works are less numerous: besides his articles, B. Mandelbrot has published general books about his work (Mandelbrot, 1975, 1977, and 1982); the books by Barnsley (1988) and Falconer (1990) both approach the mathematical aspects of the subject. Among the books treating fractals within the domain of the physical sciences are those by Feder (1988) and Vicsek (1989) (which particularly concentrates on growth phenomena), Takayasu (1990), or Le Méhauté (1990), as well as a certain number of more specialized (Avnir, 1989; Bunde and Havlin, 1991) or introductory monographs on fractals (Sapoval, 1990). More specialized reviews will be mentioned in the appropriate chapters.

1.2 The notion of dimension

A common method of measuring a length, a surface area or a volume consists in covering them with boxes whose length, surface area or volume is taken as the unit of measurement (Fig. 1.2.1). This is the principle which lies behind the use of multiple integration in calculating these quantities.

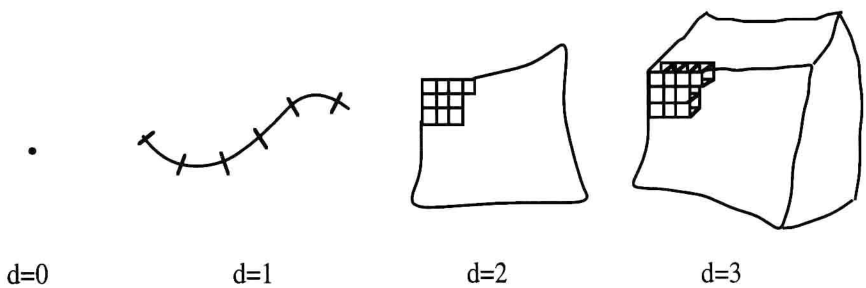

Fig. 1.2.1. Paving with lines, surfaces, or volumes.

If ε is the side (standard length) of a box and d its Euclidean dimension, the measurement obtained is

$$\mathcal{M} = N\,\varepsilon^d = N\mu,$$

where μ is the unit of measurement (length, surface area, or volume in the present case, mass in other cases). Cantor, Carathéodory, Peano, etc. showed that there exist pathological objects for which this method fails. The measurement above must then be replaced, for example, by the α-dimensional Hausdorff measure. This is what we shall now explain.

The length of the Brittany's coastline

Imagine that we would like to apply the preceding method to measure the length, between two fixed points, of a very jagged coastline such as that of

Brittany.[3] We soon notice that we are faced with a difficulty: the length $\mathcal{L}$ depends on the chosen unit of measurement ε and increases indefinitely as ε decreases (Fig. 1.2.2)!

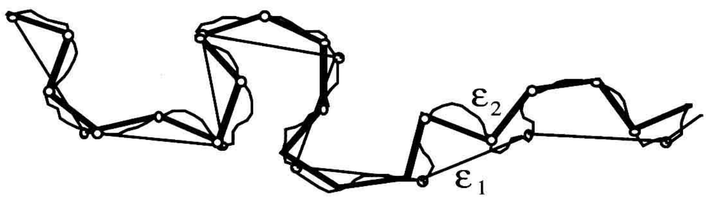

Fig. 1.2.2. Measuring the length of a coastline in relation to different units.

For a standard unit ε_1 we get a length $N_1\ \varepsilon_1$, but a smaller standard measure, ε_2, gives a new value which is larger,

$$\mathcal{L}(\varepsilon_1) = N_1\ \varepsilon_1$$
$$\mathcal{L}(\varepsilon_2) = N_2\ \varepsilon_2 \neq \mathcal{L}(\varepsilon_1)$$
$$\ldots$$

and this occurs on scales going from several tens of kilometers down to a few meters. L.F. Richardson, in 1961, studied the variations in the approximate length of various coastlines and noticed that, very generally speaking, over a

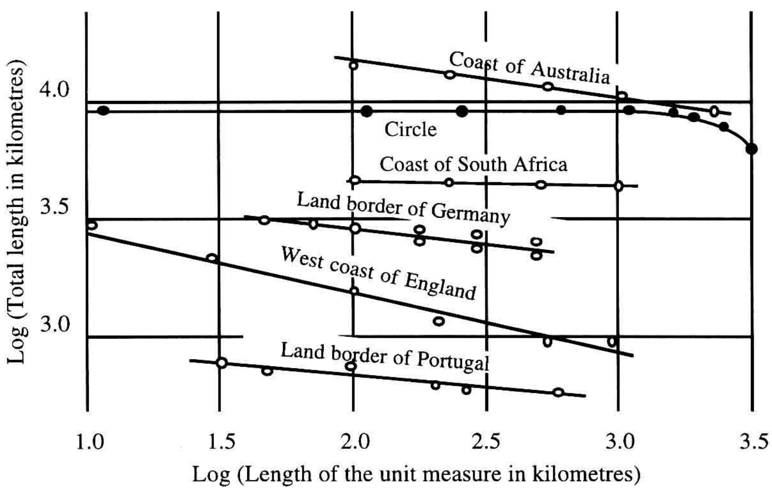

Fig. 1.2.3 Measurements of the lengths of various coastlines and land borders carried out by Richardson (1961)

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3 See the interesting preface of J. Perrin (1913) in *Atoms*, Constable (London).

large range of $\mathcal{L}(\varepsilon)$, the length follows a power law[4] in ε,

$$\mathcal{L}(\varepsilon) = N(\varepsilon)\,\varepsilon \propto \varepsilon^{-\rho}.$$

Figure 1.2.3 shows the behavior of various coastlines as functions of the unit of measurement. We can see that for a "normal" curve like the circle, the length remains constant ($\rho = 0$) when the unit of measurement becomes small enough in relation to the radius of curvature. The dimension of the circle is of course $D = 1$ (and corresponds to $\rho = 0$). The other curves display a positive exponent ρ so that their length grows indefinitely as the standard length decreases: it is impossible to give them a precise length, they are said to be nonrectifiable.[5] Moreover, these curves also prove to be nondifferentiable.

The exponent $(1+\rho)$ of $1/N(\varepsilon)$ defined above is in fact the "*fractal dimension*" as we shall see below. This method of determining the fractal size by covering the coast line with discs of radius ε is precisely the one used by Pontrjagin and Schnirelman (1932) (Mandelbrot, 1982, p. 439) to define the *covering dimension.* The idea of defining the dimension on the basis of a covering ribbon of width 2ε had already been developed by Minkowski in 1901. We shall therefore now examine these methods in greater detail.

Generally speaking, studies carried out on fractal structures rely both on those concerning nondifferentiable functions (Cantor, Poincaré, and Julia) and on those relating to the measure (dimension) of a closed set (Bouligand, Hausdorff, and Besicovitch).

1.3 Metric properties: Hausdorff dimension, topological dimension

Several definitions of fractal dimension have been proposed. These mathematical definitions are sometimes rather formal and initially not always very meaningful to the physicist. For a given fractal structure they usually give the same value for the fractal dimension, but this is not always the case. With some of these definitions, however, the calculations may prove easier or more precise than with others, or better suited to characterize a physical property.

Before giving details of the various categories of fractal structures, we shall give some mathematical definitions and various methods for calculating dimensions; for more details refer to Tricot's work (Tricot, 1988), or to Falconer's books (Falconer, 1985, 1990).

First, we remark that to define the dimension of a structure, this structure must have a notion of distance (denoted $|x-y|$) defined on it between any two of its points. This hardly poses a problem for the structures provided by nature.

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[4] The commonly used notation '$\propto$' means 'varies as': $a \propto b$ means precisely that the ratio a/b asymptotically tends towards a nonzero constant.

[5] A part of a curve is rectifiable if its length can be determined.

We should also mention that in these definitions there is always a passage to the limit $\varepsilon^{-}0$. For the actual calculation of a fractal dimension we are led to discretize (i.e., to use finite basic lengths ε): the accuracy of the calculation then depends on the relative lengths of the unit ε, and that of the system (Sec. 1.4.4).

1.3.1 The topological dimension d_T

If we are dealing with a geometric object composed of a set of points, we say that its fractal dimension is $d_T = 0$; if it is composed of line elements, $d_T = 1$, surface elements $d_T = 2$, etc.

"Composed" means here that the object is locally homeomorphic to a point, a line, a surface. The topological dimension is invariant under invertible, continuous, but not necessarily differentiable, transformations (homeomorphisms). The dimensions which we shall be speaking of are invariant under differentiable transformations (dilations).

A fractal structure possesses a fractal dimension strictly greater than its topological dimension.

1.3.2 The Hausdorff–Besicovitch dimension, or covering dimension: dim(E)

The first approach to finding the dimension of an object, E, follows the usual method of covering the object with boxes (belonging to the space in which the object is embedded) whose measurement unit $\mu = \varepsilon^{d(E)}$, where d(E) is the Euclidean dimension of the object. When d(E) is initially unknown, one possible solution takes $\mu = \varepsilon^{\alpha}$ as the unit of measurement for an unknown exponent α. Let us consider, for example, a square (d = 2) of side L, and cover it with boxes of side ε. The measure is given by $\mathcal{M} = N\mu$, where N is the number of boxes, hence $N = (L/\varepsilon)^d$. Thus,

$$\mathcal{M} = N\,\varepsilon^{\alpha} = (L/\varepsilon)^d\,\varepsilon^{\alpha} = L^2\,\varepsilon^{\alpha-2}$$

If we try $\alpha = 1$, we find that $\mathcal{M} \to \infty$ when $\varepsilon \to 0$: the "length" of a square is infinite. If we try $\alpha = 3$, we find that $\mathcal{M} \to 0$ when $\varepsilon \to 0$: the "volume" of a square is zero. The surface area of a square is obtained only when $\alpha = 2$, and its dimension is the same as that of a surface $d = \alpha = 2$.

The fact that this method can be applied for any real α is very interesting as it makes possible its generalization to noninteger dimensions.

We can formalize this measure a little more. First, as the object has no specific shape, it is not possible, in general, to cover it with identical boxes of side ε. But the object E may be covered with balls V_i whose diameter (diam V_i) is less than or equal to ε. This offers more flexibility, but requires that the