TRENDS IN MATHEMATICS

Analysis and Geometry in Several Complex Variables

Gen Komatsu
Masatake Kuranishi
Editors

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Analysis and Geometry in Several Complex Variables

Proceedings of the 40th Taniguchi Symposium

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Preface

This volume consists of a collection of articles for the proceedings of the 40th Taniguchi Symposium *Analysis and Geometry in Several Complex Variables* held in Katata, Japan, on June 23–28, 1997.

Since the inhomogeneous Cauchy-Riemann equation was introduced in the study of Complex Analysis of Several Variables, there has been strong interaction between Complex Analysis and Real Analysis, in particular, the theory of Partial Differential Equations. Problems in Complex Analysis stimulate the development of the PDE theory which subsequently can be applied to Complex Analysis. This interaction involves Differential Geometry, for instance, via the CR structure modeled on the induced structure on the boundary of a complex manifold. Such structures are naturally related to the PDE theory. Differential Geometric formalisms are efficiently used in settling problems in Complex Analysis and the results enrich the theory of Differential Geometry.

This volume focuses on the most recent developments in this interaction, including links with other fields such as Algebraic Geometry and Theoretical Physics. Written by participants in the Symposium, this volume treats various aspects of CR geometry and the Bergman kernel/ projection, together with other major subjects in modern Complex Analysis. We hope that this volume will serve as a resource for all who are interested in the new trends in this area.

We would like to express our gratitude to the Taniguchi Foundation for generous financial support and hospitality. We would also like to thank Professor Kiyosi Ito who coordinated the organization of the symposium. Finally, we greatly appreciate all the efforts of the referees.

Gen Komatsu
Masatake Kuranishi
Editors

Symposium

The 40th Taniguchi Symposium
Analysis and Geometry in Several Complex Variables
June 23–28, 1997
at Kyuzeso Seminar House, Katata, Japan

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Analysis and Geometry in Several Complex Variables

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CHAPTER I

The Bergman Kernel and a Theorem of Tian

David Catlin

Introduction

Given a domain Ω in $\mathbb{C}^n$, the Bergman kernel is the kernel of the projection operator from $L^2(\Omega)$ to the Hardy space $\mathcal{A}^2(\Omega)$. When the boundary of Ω is strictly pseudoconvex and smooth, Fefferman [2] gave a complete description of the asymptotic behavior of $K(z,z)$ as z approaches the boundary. This work was then extended by Boutet de Monvel and Sjöstrand [1] who showed that, for the same domains, a similar asymptotic expansion for $K(z,w)$ holds off the diagonal. Moreover, they showed that the Bergman kernel is a Fourier integral operator with a complex phase function. The first goal of this paper is to prove the following theorem:

Theorem 1. *Suppose E is a holomorphic vector bundle defined over a smoothly bounded strictly pseudoconvex manifold $\Omega = \{z; R(z) < 1\}$, and suppose that the L^2-norm is defined in terms of both a smooth Hermitian metric on E and a smooth metric g on the base manifold Ω. Then the Bergman kernel $K(z,w)$ of the projection onto $\mathcal{A}^2(\Omega, E)$ is a Fourier integral operator and can be represented by*

$$K(z,w) = \frac{F(z,w)}{(1-R(z,w))^{n+1}} + G(z,w)\log(1-R(z,w)). \qquad (0.1)$$

As in [1] and [2], the function $R(z,w)$ is almost analytic along the boundary diagonal. The coefficients F and G are smooth sections of the vector bundle whose fiber at (z,w) is $\mathrm{Hom}(E_w, E_z)$.

Theorem 1 is hardly a surprising result. It seems certain that the proof of Boutet de Monvel and Sjöstrand would carry over to the situation of Theorem 1 with few changes. The proof given here assumes the theorem of Boutet de Monvel-Sjöstrand and also makes use of a few simple facts about Fourier integral operators.

Secondly, we use the result of Theorem 1 to study the asymptotic behavior of a family of finite-dimensional Bergman kernels on circular domains. Let E and L be holomorphic vector bundles of rank p and 1, respectively, over a complex manifold M, and let R be a smooth Hermitian metric on L. We assume that R has been extended onto a smooth function $L \times L$ that is almost-analytic along the diagonal, linear in the first entry and anti-linear in the second.

We let $\overline{\Omega} = \{\xi \in L;\ R(\xi,\xi) \leq 1\}$, and then, using $\pi : \overline{\Omega} \longrightarrow M$, we define $\widetilde{E} = \pi^* E$ and also a metric $\widetilde{G} = \pi^* G$ on $\widetilde{E}$. Thus we obtain an L^2-norm by setting $\|\Phi\|^2 = \sum\limits_{\Omega} |\Phi|^2 \mathrm{vol}_{\tilde{g}}$, where $\tilde{g}$ is a suitably chosen metric on $T\Omega$.

Let $\mathcal{A}_d(\Omega, \widetilde{E})$ denote the space of holomorphic section of $\widetilde{E}$ on Ω that are homogeneous of order d on each fiber L_z, and let $K_d(\xi, \theta)$, having values in Hom $(\widetilde{E}_\theta, \widetilde{E}_\xi)$, denote the kernel of the projection

$$(P_d f)(\xi) = \sum_{\Omega} K_d(\xi,\theta) f(\theta) \mathrm{vol}_{\tilde{g}}(\theta) \tag{0.2}$$

of $L^2(\Omega, \widetilde{E})$ onto $\mathcal{A}_d(\Omega, \widetilde{E})$.

Theorem 2. *Suppose that the curvature of R is negative on M. Then for all $\ell = 0, 1, \dots$, there exist smooth sections $a_\ell(\xi, \theta)$ having values in Hom$(\widetilde{E}_\theta, \widetilde{E}_\xi)$ and constant along fibers of L such that*

$$K_d \sim R^d \sum_{\ell=0}^{\infty} d^{n+1-\ell} a_\ell, \tag{0.3}$$

where (0.3) means that for any integers $q,\ N \geq 0$,

$$\left\| K_d - R^d \sum_{\ell=0}^{N} d^{n+1-\ell} a_\ell \right\|_{C^q} \leq M_{N,q} d^{n+q-N}. \tag{0.4}$$

Moreover, at any point $\xi \in b\Omega$, a_0 satisfies

$$a_0(\xi,\xi) = \frac{1}{2\pi} |\lambda_1(z) \dots \lambda_n(z)|\ Id, \tag{0.5}$$

where $\lambda_1(z), \dots, \lambda_n(z)$ are the eigenvalues of the curvature form of R at $z = \pi(\xi)$.

We note that the negativity of the curvature of R means that Ω is strictly pseudoconvex. The fact that Ω is invariant under the map $\xi \longrightarrow e^{i\phi}\xi$ means that $\mathcal{A}^2(\Omega, \widetilde{E})$ is the orthogonal sum of the finite-dimensional spaces

$\mathcal{A}_d(\Omega, \widetilde{E})$. Using the kernel formula from Theorem 1 for the projection onto $\mathcal{A}^2(\Omega, \widetilde{E})$, we show that when K is written as a Taylor series in the fiber variable ζ, the only terms that act on $\mathcal{A}_d(\Omega, \widetilde{E})$ are the terms of order ζ^d. This leads to (0.3).

It is well-known that each section $\varphi \in L^2(M, E \otimes L^{*d})$ can be identified with a section $I_d(\varphi) \in L^2_d(\Omega, \widetilde{E})$, which is defined to be the set of sections in $L^2(\Omega, \widetilde{E})$ that are homogeneous of order d on each fiber L_z. Given G, R, and g, there is a naturally defined L^2-norm $\|\quad\|_{E\otimes L^{*d}}$ on $L^2(M,\ E \otimes L^{*d})$. If $\tilde{\varphi} \in L^2_d(\Omega, \widetilde{E})$, we obtain a norm $\|\tilde{\varphi}\|^2 = \|I_d^{-1}(\tilde{\varphi})\|_{E\otimes L^{*d}}$ which turns out to be a slight perturbation of the usual $\widetilde{E}$-norm. We let $K_{M,d}$ denote the kernel of the projection $P_{M,d}$ of $L^2_d(\Omega, \widetilde{E})$ onto $\mathcal{A}_d(\Omega, \widetilde{E})$ with respect to this new norm. Thus $P_{M,d}$ is just the projection onto $H^0(M,\ E \otimes L^{*d})$, transferred over to Ω.

Theorem 3. *Under the assumptions of Theorem 2, the kernel $K_{M,d}$ of $P_{M,d}$ satisfies*

$$K_{M,d} \sim \frac{2\pi}{d+2} K_d. \tag{0.6}$$

Hence $K_{M,d}$ has an asymptotic expansion of the form

$$K_{M,d} \sim R^d \sum_{\ell=0}^{\infty} A_\ell d^{n-\ell}, \tag{0.7}$$

where A_ℓ is constant along fibers and where

$$A_0(z, z) = |\lambda_1(z) \dots \lambda_n(z)|\ Id. \tag{0.8}$$

Corollary. *Let $\varphi_1 \dots, \varphi_N$ be an orthonormal basis of $H^0(M,\ E \otimes L^{*d})$ and define $B(z) = \sum_{k=0}^{N} |\varphi_k(z)|^2_{E\otimes L^{*d}}$. Then*

$$B(z) \sim \sum_{\ell=0}^{\infty} trA_\ell(z) d^{n-\ell}. \tag{0.9}$$

In the case when $E = L^*$, the above result and its corollary and also Theorem 4 which follow were obtained independently by Zelditch [5] and the author. The asymptotic description of $K_{M,d}$ in [5] is based on the study of the Szegö kernel of the disk bundle Ω.

For the final result of this paper, we use Theorem 3 to describe the asymptotic behavior of a sequence of metrics g_d introduced by Tian [4]. When the curvature of R is negative, then for large d, a basis $\Phi_1, \ldots, \Phi_{N_d}$ of $H^0(M, \; E^* \otimes L^{*d})$ leads to an embedding ϕ_d of M into the Grassmanian G_{p,N_d}. When $\Phi_1, \ldots, \Phi_{N_d}$ is an orthonormal basis, the map ϕ_d should have nice regularity properties. In particular, the pullback $g_d = \frac{1}{d}\phi_d^* g_{Gr}$ of the standard metric g_{Gr} on G_{p,N_d} can be computed. (The factor of $\frac{1}{d}$ is a normalization.)

Theorem 4. *If the Ricci curvature $Ric(R)$ is negative on M, then there are smooth $(1,1)$-forms $m_\ell, \ell = 1, 2, \ldots,$ on M such that*

$$g_d = -p \; Ric(R) + \sum_{\ell=1}^{\infty} d^{-\ell} m_\ell. \tag{0.10}$$

It follows that g_d approaches $-p \; Ric(R)$ in the C^∞ topology.

When $E = L$, this result was obtained by Tian [4] in the C^2-topology, and as noted above by Zelditch in the C^∞-topology. In [6], the Bergman projection on $L^2(\Omega, \widetilde{E})$ was used to prove an isometric embedding theorem for holomorphic vector bundles.

I would like to express my gratitude to the Taniguchi Foundation for having invited me to attend the conference in Japan last summer. I would like to thank Larry Tong for some very helpful discussions and also Betty Gick and Judy Snider for patiently typing several versions of this paper.

1. The Bergman projection

Let E be a holomorphic vector bundle over a complex manifold Ω and let $(\;\; , \;\;)$ be a Hermitian metric on E. Given a metric g on the tangent bundle, we obtain a volume form vol_g, so we can define a norm on sections of E on Ω by

$$\|F\|^2 = \int_\Omega (F, F) \; \mathrm{vol}_g. \tag{1.1}$$

If $\Phi_\nu, \; \nu = 1, 2, \ldots$ is an orthonormal basis of $\mathcal{A}^2(\Omega, E)$, the set of holomorphic sections of E, then the Bergman projection $P \colon L^2(\Omega, E) \to \mathcal{A}^2(\Omega, E)$ can be written as $(PF)(z) = \sum_{\nu=1}^\infty \langle F, \Phi_\nu \rangle \Phi_\nu(z)$.

In order to describe the kernel of P, let $e_1', \ldots, e_p'$ and $e_1'', \ldots, e_p''$ be frames for E in neighborhoods U and V, respectively. We let F_U, $(PF)_V$, etc., denote the column vector of coefficients of F and PF with respect to $e_1', \ldots, e_p'$ and $e_1'', \ldots, e_p''$. If we define a $p \times p$ matrix $A_U(w)$ by $[A_{jk}(w)] =$

$(e_k(w), e_j(w))$, and if F is supported in U, then we obtain

$$(PF)_V(z) = \int_\Omega \sum_\nu \Phi_{\nu,V}(z)\Phi^*_{\nu,U}(w)A_U(w)F_U(w)\ \mathrm{vol}_g(w).$$

Thus the kernel $K(z,w)$ defined by

$$(PF)(z) = \int_\Omega K(z,w)F(w)\ \mathrm{vol}_g$$

takes values in Hom (E_w, E_z), and the local representation of K in $V \times U$ is

$$K_{V,U}(z,w) = \sum_\nu \Phi_{\nu,V}(z)\Phi^*_{\nu,U}(w)A_U(w).$$

Moreover, it follows immediately that the usual property that K is holomorphic in z and anti-holomorphic in w becomes $K_{V,U}(z,w)A_U^{-1}(w)$ is holomorphic in z and anti-holomorphic in w.

Closely related to the Bergman kernel is the quantity $B(z) = \sum_\nu |\Phi_\nu(z)|^2$, which, relative to the frame $e_1, \ldots, e_p$ in U, equals $\sum_\nu \Phi^*_{\nu,U}(z)A_U(z)\Phi_{\nu,U}(z)$. By taking the trace, we see that

$$\mathrm{tr}\ (\Phi_{\nu,U}(z)\Phi^*_{\nu,U}(z)A_U(z)) = \Phi^*_{\nu,U}A_U(z)\Phi_{\nu,U}(z),$$

so that by summing over ν, we obtain

$$\mathrm{tr}\ K(z,z) = B(z). \tag{1.2}$$

Our goal is to show that $K(z,w)$ can be written as a Fourier integral operator, just as in the well known case when E is the trivial bundle and $\langle\ ,\ \rangle$ and g are the standard metrics, as proved by Boutet de Monvel and Sjöstrand in [1].

We first consider the local problem and assume that D is a strictly pseudoconvex domain in $\mathbb{C}^n$ and suppose that smooth sections $e_1, \ldots, e_p$ of a vector bundle E are defined on $\overline{D}$. We assume that $(\ ,\ )$ is a metric defined on E and we define a matrix A' by $A'_{jk}(z) = (e_k(z),\ e_j(z))$. If in addition there is a volume element $\mathrm{vol}(z) = b(z)\ \mathrm{vol}_0$, where vol_0 is the Euclidean volume element in $\mathbb{C}^n$, then a global norm on sections of E is given by $\|F\|^2 = \int_D F^*AF\ \mathrm{vol}_0$, where $A = A'b$, and where in the integral F denotes the column vector given by the coefficients $F_1, \ldots, F_p$ of $F = \sum_{k=1}^n F_k e_k$.