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TOPOLOGICAL DEGREE THEORY AND APPLICATIONS

Donal O'Regan, Yeol Je Cho, and Yu-Qing Chen

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SERIES IN MATHEMATICAL ANALYSIS AND APPLICATIONS

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Chapter 1

BROUWER DEGREE THEORY

Let R be the real numbers, $R^n = \{x = (x_1, x_2, \cdots, x_n) : x_i \in R \text{ for } i = 1, 2, \cdots, n\}$ with $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and let $\Omega \subset R^n$, and let $f : \Omega \to R^n$ be a continuous function. A basic mathematical problem is: Does $f(x) = 0$ have a solution in Ω? It is also of interest to know how many solutions are distributed in Ω. In this chapter, we will present a number, the topological degree of f with respect to Ω and 0, which is very useful in answering these questions. To motivate the process, let us first recall the winding number of plane curves, a basic topic in an elementary course in complex analysis. Let C be the set of complex numbers, $\Gamma \subset C$ an oriented closed C^1 curve and $a \in C \setminus \Gamma$. Then the integer

$$w(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - a} dz \tag{1}$$

is called the winding number of Γ with respect to $a \in C \setminus \Gamma$. Now, let $G \subset C$ be a simply connected region and $f : G \to C$ be analytic and $\Gamma \subset G$ a closed C^1 curve such that $f(z) \neq 0$ on Γ. Then we have

$$w(f(\Gamma), 0) = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{1}{z} dz = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = \sum_i w(\Gamma, z_i)\alpha_i, \tag{2}$$

where z_i are the zeros of f in the region enclosed by Γ and α_i are the corresponding multiplicites. If we assume in addition that Γ has positive orientation and no intersection points, then we know from Jordan's Theorem, which will be proved later in this chapter, that $w(\Gamma, z_i) = 1$ for all z_i. Thus (2) becomes

$$w(f(\Gamma), 0) = \sum_i \alpha_i. \tag{3}$$

So we may say that f has at least $|w(f(\Gamma), 0)|$ zeros in G. The winding number is a very old concept which goes back to Cauchy and Gauss. Kronecker, Hadamard, Poincare, and others extended formula (1). In 1912, Brouwer [32] introduced the so-called Brouwer degree in R^n (see Browder [35], Sieberg [277] for historical developments). In this chapter, we introduce the Brouwer degree theory and its generalization to functions in VMO. This chapter is organized as follows:

In Section 1.1 we introduce the notion of a critical point for a differentiable function f. We then prove Sard's Lemma, which states that the set of critical points of a C^1 function is "small". Our final result in this section shows how a continuous function can be approximated by a C^∞ function.

In Section 1.2 we begin by defining the degree of a C^1 function using the Jacobian. Also we present an integral representation which we use to define the degree of a continuous function. Also in this section we present some properties of our degree (see theorems 1.2.6, 1.2.12, and 1.2.13) and some useful consequences. For example, we prove Brouwer's and Borsuk's fixed point theorem, Jordan's separation theorem and an open mapping theorem. In addition we discuss the relation between the winding number and the degree.

In Section 1.3 we discuss some properties of the average value function and then we introduce the degree for functions in VMO.

In Section 1.4 we use the degree theory in Section 1.2 to present some existence results for the periodic and anti-periodic first order ordinary differential equations.

1.1 Continuous and Differentiable Functions

We begin with the following Bolzano's intermediate value theorem:

Theorem 1.1.1. Let $f : [a,b] \to R$ be a continuous function, then, for m between $f(a)$ and $f(b)$, there exists $x_0 \in [a,b]$ such that $f(x_0) = m$.

Corollary 1.1.2. Let $f : [a,b] \to R$ be a continuous function such that $f(a)f(b) < 0$. Then there exists $x_0 \in (a,b)$ such that $f(x_0) = 0$.

Corollary 1.1.3. Let $f : [a,b] \to [a,b]$ be a continuous function. Then there exists $x_0 \in [a,b]$ such that $f(x_0) = x_0$.

Let $\Omega \subset R^n$ be an open subset. We recall that a function $f : \Omega \to R^n$ is *differentiable* at $x_0 \in \Omega$ if there is a matrix $f'(x_0)$ such that $f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$, where $x_0 + h \in \Omega$ and $\frac{|o(h)|}{|h|}$ tends to zero as $|h| \to 0$.

We use $C^k(\Omega)$ to denote the space of k-times continuously differentiable functions. If f is differentiable at x_0, we call $J_f(x_0) = det f'(x_0)$ the Jacobian of f at x_0. If $J_f(x_0) = 0$, then x_0 is said to be a critical point of f and we use $S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}$ to denote the set of critical points of f, in Ω. If $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, then y is said to be a regular value of f. Otherwise, y is said to be a singular value of f.

Lemma 1.1.4. (Sard's Lemma) Let $\Omega \subset R^n$ be open and $f \in C^1(\Omega)$. Then $\mu_n(f(S_f(\Omega)) = 0$, where μ_n is the n-dimensional Lebesgue measure.

Proof. Since Ω is open, $\Omega = \cup_{i=1}^{\infty} Q_i$, where Q_i is a cube for $i = 1, 2, \cdots$. We only need to show that $\mu_n(f(S_f(Q))) = 0$ for a cube $Q \subset \Omega$. In fact, let

l be the lateral length of Q. By the uniform continuity of f' on Q, for any given $\epsilon > 0$, there exists an integer $m > 0$ such that

$$|f'(x) - f'(y)| \leq \epsilon$$

for all $x, y \in Q$ with $|x - y| \leq \frac{\sqrt{n}l}{m}$. Therefore, we have

$$\begin{aligned}|f(x) - f(y) - f'(y)(x-y)| &\leq \int_0^1 |f'(y + t(x-y)) - f'(y)||x-y|dt \\ &\leq \epsilon|x-y|\end{aligned}$$

for all $x, y \in Q$ with $|x - y| \leq \frac{\sqrt{n}l}{m}$. We decompose Q into r cubes, Q^i, of diameter $\frac{\sqrt{n}l}{m}$, $i = 1, 2, \cdots, r$. Since $\frac{l}{m}$ is the lateral length of Q^i, we have $r = m^n$. Now, suppose that $Q^i \cap S_f(\Omega) \neq \emptyset$. Choosing $y \in Q^i \cap S_f(\Omega)$, we have $f(y+x) - f(y) = f'(y)x + R(y,x)$ for all $x \in Q^i - y$, where $|R(y, x+y)| \leq \epsilon\frac{\sqrt{n}l}{m}$. Therefore, we have

$$f(Q^i) = f(y) + f'(y)(Q^i - y) + R(y, Q^i).$$

But $f'(y) = 0$, so $f'(y)(Q^i - y)$ is contained in an $(n-1)$- dimensional subspace of R^n. Thus, $\mu_n(f'(y)(Q^i - y)) = 0$, so we have

$$\mu_n(f(Q^i)) \leq 2^n\epsilon^n(\frac{\sqrt{n}l}{m})^n.$$

Obviously, $f(S_f(Q)) \subset \cup_{i=1}^r f(Q^i)$, so we have

$$\mu_n(f(S_f(Q)) \leq r2^n\epsilon^n(\frac{\sqrt{n}l}{m})^n = 2^n\epsilon^n(\sqrt{n}l)^n.$$

By letting $\epsilon \to 0^+$, we obtain $\mu_n(f(S_f(Q))) = 0$. Therefore, $\mu_n(f(S_f(\Omega))) = 0$. This completes the proof.

Proposition 1.1.5. Let $K \subset R^n$ be a bounded closed subset, and $f : K \to R^n$ continuous. Then there exists a continuous function $\tilde{f} : R^n \to \overline{conv f(K)}$ such that $\tilde{f}(x) = f(x)$ for all $x \in K$, where $conv f(K)$ is the convex hull of $f(K)$.

Proof. Since K is bounded closed subset, there exists at most countable $\{k_i : i = 1, 2, \cdots\} \subset K$ such that $\overline{\{k_i : i = 1, 2, \cdots\}} = K$. Put

$$d(x, K) = \inf_{y \in K} |x - y|, \quad \alpha_i(x) = \max\{2 - \frac{|x - k_i|}{d(x, A)}, 0\}$$

for any $x \notin K$ and

$$\tilde{f}(x) = \begin{cases} f(x), & x \in K, \\ \frac{\sum_{i \geq 1} 2^{-i}\alpha_i(x)f(k_i)}{\sum_{i\geq 1} 2^{-i}\alpha_i(x)}, & x \notin K. \end{cases}$$

Then $\tilde{f}$ is the desired function.

Proposition 1.1.6. Let $K \subset R^n$ be a bounded closed subset and $f : K \to R^n$ continuous. Then there exists a function $g \in C^\infty(R^n)$ such that $|f(x) - g(x)| < \epsilon$.

Proof. By Proposition 1.1.5, there exists a continuous extension $\tilde{f}$ of f to R^n. Define the following function

$$\phi(x) = \begin{cases} ce^{-\frac{1}{1-|x|}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{1.1}$$

where c satisfies $\int_{R^n} \phi(x)dx = 1$. Set $\phi_\lambda(x) = \lambda^{-n}\phi(\frac{x}{\lambda})$ for all $x \in R^n$ and

$$f_\lambda(x) = \int_{R^n} \tilde{f}(y)\phi_\lambda(y - x)dx \text{ for all } x \in R^n,\ \lambda > 0.$$

It is obvious that $supp f_\lambda = \overline{\{x \in R^n : f_\lambda(x) \neq 0\}} = \{x : |x| \leq \lambda\}$ for all $\lambda > 0$. Consequently, we have $f_\lambda \in C^\infty$ and $f_\lambda(x) \to f(x)$ uniformly on K as $\lambda \to 0^+$. Taking g as f_λ for sufficiently small λ, g is the desired function. This completes the proof.

1.2 Construction of Brouwer Degree

Now, we give the construction of Brouwer degree in this section as follows:

Definition 1.2.1. Let $\Omega \subset R^N$ be open and bounded and $f \in C^1(\overline{\Omega})$. If $p \notin f(\partial\Omega)$ and $J_f(p) \neq 0$, then we define

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn J_f(x),$$

where $deg(f, \Omega, p) = 0$ if $f^{-1}(p) = \emptyset$.

The next result gives another equivalent form of Definition 1.2.1.

Proposition 1.2.2. Let Ω, f and p be as in Definition 1.2.1 and let

$$\phi_\epsilon(x) = \begin{cases} c\epsilon^{-n} e^{-\frac{1}{1-|\epsilon^{-1}x|^2}}, & |x| < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

where c is a constant such that $\int_{R^n} \phi(x) = 1$. Then there exists $\epsilon_0 = \epsilon_0(p, f)$ such that

$$deg(f, \Omega, p) = \int_\Omega \phi_\epsilon(f(x) - p)J_f(x)dx \quad \text{for all } \epsilon \in (0, \epsilon_0).$$

Proof. The case $f^{-1}(p) = \emptyset$ is obvious. Assume that

$$f^{-1}(p) = \{x_1, x_2, \cdots, x_n\}.$$

We can find disjoint balls $B_r(x_i)$ and a neighborhood V_i of p such that $f : B_r(x_i) \to V_i$ is a homeomorphism and $sgnJ_f(x) = sgnJ_f(x_i)$ in $B_i(x_i)$. We may take $r_0 > 0$ such that $B_{r_0}(p) \subset \cap_{i=1}^n V_i$ and set $U_i = B_r(x_i) \cap f^{-1}(B_{r_0}(p))$. Then $|f(x) - p| \geq \delta$ on $\overline{\Omega} \setminus \cup_{i=1}^n U_i$ for some $\delta > 0$ and so, for any $\epsilon < \delta$, we have

$$\int_\Omega \phi_\epsilon(f(x) - p) J_f(x) dx = \sum_{i=1}^n sgnJ_f(x_i) \int_{U_i} \phi_\epsilon(f(x) - p)|J_f(x)| dx.$$

But we have

$$\begin{aligned} &J_f(x) = J_{f-p}(x), \\ &\int_{U_i} \phi_\epsilon(f(x) - p)|J_f(x)| dx = \int_{B_{r_0}} \phi_\epsilon(x) dx = 1, \\ &\epsilon < \min\{r_0, \delta\}. \end{aligned}$$

This completes the proof.

Definition 1.2.3. Let $\Omega \subset R^N$ be open and bounded and $f \in C^2(\overline{\Omega})$. If $p \notin f(\partial\Omega)$. Then we define

$$deg(f, \Omega, p) = deg(f, \Omega, p'),$$

where p' is any regular value of f that $|p' - p| < d(p, f(\partial\Omega))$.

We need to check that, for any two regular values p_1 and p_2 of f,

$$deg(f, \Omega, p_1) = deg(f, \Omega, p_2).$$

For any $\epsilon < d(p, f(\partial\Omega)) - \max\{|p - p_i| : i = 1, 2\}$, we have

$$deg(f, \Omega, p_i) = \int_\Omega \phi_\epsilon(f(x) - p_i) J_f(x) dx \quad \text{for } i = 1, 2.$$

Notice that

$$\phi_\epsilon(x - p_2) - \phi_\epsilon(x - p_1) = divw(x),$$

where

$$w(x) = (p_1 - p_2) \int_0^1 \phi_\epsilon(x - p_1 + t(p_1 - p_2)) dt.$$

We show that there exists a function $v \in C^1(R^N)$ such that $supp(v) \subset \Omega$ and

$$[\phi_\epsilon(f(x) - p_2) - \phi_\epsilon(f(x) - p_1)] J_f(x) = divv(x) \text{ for all } x \in \Omega.$$

Lemma 1.2.4. Let $\Omega \subset R^N$ be open, $f \in C^2(\overline{\Omega})$ and let d_{ij} be the cofactor of $\frac{\partial f_j}{\partial x_i}$ in $J_f(x)$ and

$$v_i(x) = \begin{cases} \sum_{j=1}^N w_j(f(x))d_{ij}(x) & x \in \overline{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(v_1(x), v_1(x), \cdots, v_N(x))$ satisfies $divv(x) = divw(f(x))J_f(x)$.

Proof. Since $supp(w) \subset \overline{B(p,r)}$ for $r \leq max\{|p - p_i| : i = 1,2\} + \epsilon < d(p, \partial\Omega)$, we have

$$supp(v) \subset \Omega,$$

$$\partial_i v_i(x) = \sum_{j,k=1}^N d_{jk}\partial_k W_j(f(x))\partial_i f_k(x) + \sum_{j=1}^N W_j(f(x))\partial_i d_{ij}(x),$$

where $\partial_k = \frac{\partial}{\partial x_k}$. Now, we claim that

$$\sum_{i=1}^N \partial_i d_{ij}(x) = 0 \quad \text{for } j = 1, 2, \cdots, N.$$

For any given j, let f_{x_k} denote the column

$$(\partial_k f_1, \cdots, \partial_k f_{j-1}, \partial_k f_{j+1}, \cdots, \partial_k f_n).$$

Then we have

$$d_{ij}(x) = (-1)^{i+j} det(f_{x_1}, \cdots, f_{i-1}, f_{i+1}, \cdots, f_N).$$

Therefore, it follows that

$$\partial_i d_{ij}(x) = (-1)^{i+j} \sum_{k=1}^N det(f_{x_1}, \cdots, f_{x_{i-1}}, f_{x_{i+1}}, \cdots, \partial_i f_{x_k}, \cdots, f_{x_N}).$$

Set

$$a_{ki} = det(\partial_i f_{x_k}, f_{x_1}, \cdots, f_{x_{i-1}}, f_{x_{i+1}}, \cdots, f_{x_{k-1}}, f_{x_{k+1}}, \cdots, f_{x_N}),$$

then we have $a_{ki} = a_{ik}$ and

$$\begin{aligned}(-1)^{i+j}\partial_i d_{ij}(x) &= \sum_{i,k=1}^N (-1)^{k-1} a_{ki} + \sum_{k>i} (-1)^{k-2} a_{ki} \\ &= \sum_{k=1}^N (-1)^{k-1} \delta_{ki} a_{ki},\end{aligned}$$

where $\delta_{ki} = 1$ for $k < i$, $\delta_{ii} = 0$ and $\delta_{ki} = -\delta_{ik}$ for $i, k = 1, 2, \cdots, N$. Hence we have

$$(-1)^j \sum_{i=1}^{N} \partial_i d_{ij}(x) = \sum_{i,k=1}^{N} (-1)^{k-1+i} \gamma_{ki} a_{ki} = \sum_{k,i=1}^{N} (-1)^{i-1+k} \gamma_{ik} a_{ik}$$
$$= - \sum_{i,k=1}^{N} (-1)^{k-1+i} \gamma_{ki} a_{ki} = 0.$$

Now, we have

$$\partial_i v_i(x) = \sum_{j,k=1}^{N} d_{i,j} \partial_k w_j(f(x)) \partial_i f_k(x) + \sum_{j=1}^{N} w_j(f(x)) \partial_i d_{ij}(x).$$

On the other hand, $\sum_{i=1}^{N} d_{ij} \partial_i f_k(x) = \delta_{jk} J_f(x)$ with Kronecker's δ_{jk}. Therefore, it follows that

$$divv(x) = \sum_{k,j=1}^{N} \partial_k w_j(f(x)) \delta_{jk} J_f(x) = divw(f(x)) J_f(x).$$

This completes the proof.

Finally, we are ready to introduce the following definition:

Definition 1.2.5. Let $\Omega \subset R^N$ be open and bounded, $f \in C(\overline{\Omega})$ and $p \notin f(\partial\Omega)$. Then we define

$$deg(f, \Omega, p) = deg(g, \Omega, p),$$

where $g \in C^2(\overline{\Omega})$ and $|g - f| < d(p, f(\partial\Omega))$.

Now, one may check the following properties by a reduction to the regular case.

Theorem 1.2.6. Let $\Omega \subset R^N$ be an open bounded subset and $f : \overline{\Omega} \to R^N$ be a continuous mapping. If $p \notin f(\partial\Omega)$, then there exists an integer $deg(f, \Omega, p)$ satisfying the following properties:

(1) (*Normality*) $deg(I, \Omega, p) = 1$ if and only if $p \in \Omega$, where I denotes the identity mapping;

(2) (*Solvability*) If $deg(f, \Omega, p) \neq 0$, then $f(x) = p$ has a solution in Ω;

(3) (*Homotopy*) If $f_t(x) : [0,1] \times \overline{\Omega} \to R^N$ is continuous and $p \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$, then $deg(f_t, \Omega, p)$ does not depend on $t \in [0, 1]$;

(4) (*Additivity*) Suppose that Ω_1, Ω_2 are two disjoint open subsets of Ω and $p \notin f(\overline{\Omega} - \Omega_1 \cup \Omega_2)$. Then $deg(f, \Omega, p) = deg(f, \Omega_1, p) + deg(f, \Omega_2, p)$;

(5) $deg(f, \Omega, p)$ is a constant on any connected component of $R^n \setminus f(\partial\Omega)$.

As consequences of Theorem 1.2.6, we have the following results:

Theorem 1.2.7. Let $f : \overline{B(0,R)} \subset R^n \to \overline{B(0,R)}$ be a continuous mapping. If $|f(x)| \leq R$ for all $x \in \partial B(0,R)$, then f has a fixed point in $\overline{B(0,R)}$.

Proof. We may assume that $x \neq f(x)$ for all $x \in \partial B(0,R)$. Put $H(t,x) = x - tf(x)$ for all $(t,x) \in [0,1] \times \overline{B(0,R)}$. Then $0 \neq H(t,x)$ for all $[0,1] \times \partial B(0,R)$. Therefore, we have

$$deg(I - f, B(0,R), 0) = deg(I, B(0,R), 0) = 1.$$

Hence f has a fixed point in $\overline{B(0,R)}$. This completes the proof.

From Theorem 1.2.7, we have the well-known Brouwer fixed point theorem:

Theorem 1.2.8. Let $C \subset R^n$ be a nonempty bounded closed convex subset and $f : C \to C$ be a continuous mapping. Then f has a fixed point in C.

Proof. Take $B(0,R)$ such that $C \subset B(0,R)$ and let $r : \overline{B(0,R)} \to C$ be a retraction. By Theorem 1.2.7, there exists $x_0 \in \overline{B(0,R)}$ such that $frx_0 = x_0$. Therefore, $x_0 \in C$, and so we have $rx_0 = x_0$. This completes the proof.

Theorem 1.2.9. Let $f : R^n \to R^n$ be a continuous mapping and $0 \in \Omega \subset R^n$ with Ω an open bounded subset. If $(f(x), x) > 0$ for all $x \in \partial\Omega$, then $deg(f, \Omega, 0) = 1$.

Proof. Put $H(t,x) = tx + (1-t)f(x)$ for all $(t,x) \in [0,1] \times \overline{\Omega}$. Then $0 \notin H([0,1] \times \partial\Omega)$, and so we have

$$deg(f, \Omega, 0) = deg(I, \Omega, 0) = 1.$$

This completes the proof.

Corollary 1.2.10. Let $f : R^n \to R^n$ be a continuous mapping. If

$$\lim_{|x| \to \infty} \frac{(f(x), x)}{|x|} = +\infty,$$

then $f(R^n) = R^n$.

Proof. For any $p \in R^n$, it is easy to see that there exists $R > 0$ such that $(f(x) - p, x) > 0$ for all $x \in \partial B(0,R)$, where $B(0,R)$ is the open ball centered at zero with radius R. By Theorem 1.2.9, we have

$$deg(f - p, B(0,R), 0) = 1$$

and so $f(x) - p = 0$ has a solution in $B(0,R)$. This completes the proof.

Theorem 1.2.11. (**Borsuk's Theorem**) Let $\Omega \subset R^n$ be open bounded and symmetric with $0 \in \Omega$. If $f \in C(\overline{\Omega})$ is odd and $0 \notin f(\partial\Omega)$, then $d(f, \Omega, 0)$ is odd.

Proof. Without loss of generality, we may assume that $f \in C^1(\overline{\Omega})$ with $J_f(0) \neq 0$. Next, we define a mapping $g \in C^1(\overline{\Omega})$ sufficiently close to f by induction as follows:

Let $\phi \in C^1(R)$ be an odd mapping with $\phi'(0) = 0$ and $\phi(t) = 0$ if and only if $t = 0$. Put $\Omega_k = \{x \in \Omega : x_k \neq 0\}$ and $h(x) = \frac{f(x)}{\phi(x_1)}$ for all $x \in \Omega_1$. Choose $|y_1|$ sufficiently small such that y_1 is a regular value for h on Ω_1. Put $g_1(x) = f(x) - \phi(x_1)y_1$, then 0 is a regular value for g_1 on Ω_1.

Suppose that we have already an odd $g_k \in C^1(\overline{\Omega})$ close to f such that 0 is a regular value for g_k on Ω_k. Then we define $g_{k+1}(x) = g_k(x) - \phi(x_{k+1})y_{k+1}$ with $|y_{k+1}|$ small enough such that 0 is a regular value for g_{k+1} on Ω_{k+1}.

If $x \in \Omega_{k+1}$ and $x_{k+1} = 0$, then

$$x \in \Omega_k, \quad g_{k+1}(x) = g_k(x), \quad g'_{k+1}(x) = g'_k(x)$$

and hence $J_{g_{k+1}}(x) \neq 0$. By induction, we also have $g'_n(0) = g'_1(0) = f'(0)$ and so 0 is a regular value for g_n. By Definition 1.2.5 and Definition 1.2.1, we know that

$$deg(f, \Omega, 0) = deg(g_n, \Omega, 0) = sgnJ_{g_n}(0) + \sum_{x \in g^{-1}(0), x \neq 0} sgnJ_{g_n}(x)$$

and thus $deg(f, \Omega, 0)$ is odd. This completes the proof.

The following theorem shows the relationship between Brouwer degrees in different dimensional spaces:

Theorem 1.2.12. Let $\Omega \subset R^n$ be an open bounded subset, $1 \leq m < n$, let $f : \overline{\Omega} \to R^m$ be a continuous function and let $g = I - f$. If $y \notin (I - f)(\partial\Omega)$, then

$$deg(g, \Omega, y) = deg(g_m, \Omega \cap R^m, y),$$

where g_m is the restriction of g on $\overline{\Omega} \cap R^m$.

Proof. We may assume that $f \in C^2(\overline{\Omega})$ and y is a regular value for g on $\overline{\Omega}$. A direct computation yields that $J_g(x) = J_{g_m}(x)$ and so the conclusion follows from Definition 1.2.1. This completes the proof.

Let $\Omega \subset R^n$ be open and bounded and let $f \in C(\overline{\Omega})$. By the homotopy invariance of $deg(f, \Omega, y)$, we know that $deg(f, \Omega, y)$ is the same integer as y ranges through the same connected component U of $R^n \setminus f(\partial\Omega)$. Therefore, it is reasonable to denote this integer by $deg(f, \Omega, U)$. The unbounded connected component is denoted by U_∞. Now, we have the product formula:

Theorem 1.2.13. Let $\Omega \subset R^n$ be an open bounded subset, $f \in C(\overline{\Omega})$, $g \in C(R^n)$ and let U_i be the bounded connected components of $R^n \setminus f(\partial\Omega)$.

If $p \notin (gf)(\partial\Omega)$, then

$$deg(gf, \Omega, p) = \sum_i deg(f, \Omega, U_i) deg(g, U_i, p), \quad (1.2.1)$$

where only finitely many terms are not zero.

Proof. We first prove (1.2.1) only has finitely many non-zero terms. Take $r > 0$ such that $f(\overline{\Omega}) \subset B_r(0)$. Then it follows that $M = \overline{B_r(0)} \cap g^{-1}(p)$ is compact, $M \subset R^n \setminus f(\partial\Omega) = \cup_{i\geq 1} U_i$ and there exists finitely many i, say $i = 1, 2, \cdots, t$, such that $\cup_{i=1}^{t+1} U_i \supseteq M$, where $U_{t+1} = U_\infty \cap B_{r+1}$. We have

$$deg(f, \Omega, U_{t+1}) = 0, \quad deg(g, U_i, p) = 0$$

for $i \geq t+2$ since $U_j \subset B_r(0)$ and $g^{-1}(y) \cap U_j = \emptyset$ for $j \geq t+2$. Therefore, the right side of (1.2.1) has only finitely many terms different from zero.

We first suppose that $f \in C^1(\overline{\Omega}), g \in C^1(R^n)$ and p is a regular value of gf, so we have

$$deg(gf, \Omega, p) = \sum_{x \in (gf)^{-1}(p)} sgn J_{gf}(x) = \sum_{x \in (gf)^{-1}(p)} sgn J_g(f(x)) sgn J_f(x)$$

and note

$$\begin{aligned}
&\sum_{x \in f^{-1}(z), z \in g^{-1}(p)} sgn J_g(z) sgn J_f(x) \\
&= \sum_{z \in g^{-1}(p), z \in f(\Omega)} sgn J_g(z) [\sum_{x \in f^{-1}(z)} sgn J_f(x)] \\
&= \sum_{z \in f(\Omega), g(z) = p} sgn J_g(z) deg(f, \Omega, z) \\
&= \sum_{i=1}^{t} \sum_{z \in U_i} sgn J_g(z) deg(f, \Omega, z) \\
&= \sum_i deg(f, \Omega, U_i) deg(g, U_i, p).
\end{aligned}$$

For the general case $f \in C(\overline{\Omega})$ and $g \in C(R^n)$, Put

$$V_m = \{z \in B_{r+1}(0) \setminus f(\partial\Omega) : deg(f, \Omega, z) = m\},$$

$$N_m = \{i \in N : deg(f, \Omega, U_i) = m\}.$$

Obviously, $V_m = \cup_{i \in N_m} U_i$ and thus we have

$$\sum_i deg(f, \Omega, U_i) deg(g, U_i, p) = \sum_m [\sum_{i \in N_m} deg(g, U_i, p)] = \sum_m deg(g, V_m, p).$$