Ralph Martin
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LNCS 4647

Mathematics of Surfaces XII

12th IMA International Conference
Sheffield, UK, September 2007
Proceedings

Springer

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Library of Congress Control Number: 2007931446

CR Subject Classification (1998): I.3.5, I.3, I.1, J.6, G.1, F.2, I.4

LNCS Sublibrary: SL 1 – Theoretical Computer Science and General Issues

ISSN 0302-9743
ISBN-10 3-540-73842-8 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-73842-8 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springer.com

Printed in Germany

Typesetting: Camera-ready by author, data conversion by Scientific Publishing Services, Chennai, India
Printed on acid-free paper SPIN: 12097791 06/3180 5 4 3 2 1 0

Lecture Notes in Computer Science 4647

Preface

This volume collects the papers accepted for presentation at the 12th IMA Conference on the Mathematics of Surfaces, held at Ranmoor Hall, Sheffield, UK, September, 4–6, 2007. Contributors to this volume include authors from many countries in America, Asia, and Europe. The papers presented here reflect the applicability of various aspects of mathematics to engineering and computer science, especially in domains such as computer-aided design, computer vision, and computer graphics.

The papers in the present volume include eight invited papers as well as a larger number of submitted papers. They cover a range of ideas from underlying theoretical tools to industrial uses of surfaces. Surface types considered range from meshes to parametric and implicit surfaces; some papers investigate general classes of surfaces while others focus more specifically on surfaces such as developable surfaces and Dupin's cyclides. Research is reported on theoretical aspects of surfaces including topology, parameterization, differential geometry, and conformal geometry, and also more practical topics such as geometric tolerances, computing shape from shading, and medial axes for industrial applications. Other specific areas of interest include subdivision schemes, solutions of differential equations on surfaces, knot insertion, surface segmentation, surface deformation, and surface fitting.

We would like to thank all those who attended the conference and helped to make it a success. We are particularly grateful to Lucy Nye at the Institute of Mathematics and its Applications for her hard work in organizing many aspects of the conference, and to Anna Kramer and Frank Holzwarth of Springer for their help in publishing this volume. Following this preface is a list of distinguished researchers who formed the International Programme Committee, and who freely gave their time in helping to assess papers for these proceedings. Due to their work, many of the papers have been considerably improved. Our thanks go to all of them, and to other people upon whom they called to help with refereeing.

June 2007

Ralph Martin
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Table of Contents

Regularity Criteria for the Topology of Algebraic Curves and Surfaces

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Abstract. In this paper, we consider the problem of analysing the shape of an object defined by polynomial equations in a domain. We describe regularity criteria which allow us to determine the topology of the implicit object in a box from information on the boundary of this box. Such criteria are given for planar and space algebraic curves and for algebraic surfaces. These tests are used in subdivision methods in order to produce a polygonal approximation of the algebraic curves or surfaces, even if it contains singular points. We exploit the representation of polynomials in Bernstein basis to check these criteria and to compute the intersection of edges or facets of the box with these curves or surfaces. Our treatment of singularities exploits results from singularity theory such as an explicit Whitney stratification or the local conic structure around singularities. A few examples illustrate the behavior of the algorithms.

1 Introduction

In this paper, we consider the problem of analysing the shape of an object defined by polynomial equations on a bounded domain. Such a problem appears naturally when one has to compute with (algebraic) implicit surfaces [1], but also in algorithms on parameterised curves and surfaces. Typically computing the intersection of two parameterised surfaces leads to the problem of describing or analysing an implicit curve in a 4-dimensional space [2,3].

Our aim is to describe subdivision methods, which given input equations defining such an implicit object, compute a linear approximation of this object, with the same *topology*. The field of application of such methods is Geometric Modeling, where the (semi)-algebraic models used to represent shapes are considered approximations of the *real* geometry. That is, either their coefficients are known with some error or the model itself is an approximation of the actual geometry. In this modelisation process, it is assumed that making the error tend to 0, the representation converges to the actual geometry, at least conceptually. We are going to follow this line, with two specific objectives in mind:

- provide guarantees if possible.
- adapt the computation to the local difficulties of the problem.

Several methods exist to visualize or to mesh a (smooth) implicit surface. Ray tracing techniques [4] which compute the intersection between the ray from the

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eye of the observer and the first object of the scene, produce very nice static views of these surfaces. However isolated singular curves are not well treated and the output of such methods is an image, not a mesh that can be used for other computation.

The famous 'marching cube' method [5,6] developed in order to reconstruct images in 3 dimensions starting from medical data, is based on the construction of grids of values for the function and of sign analysis. It is not adaptive to the geometry of the shape, gives no guarantee of correctness and applies only to smooth surfaces.

Marching polygonizer methods improve the adaptivity of the marching cube by computing only the 'useful' cells [7,8,9], that is those which cut the surface. The algorithm starts from a valid cube (or tetrahedron), and propagate towards the connected cells, which cut the surface. Other variants of the Marching Cube approach have been proposed, to adapt to the geometry of the surface but still with a large number of voxels, even in regions where the surface is very regular. Moreover, the treatment of singularities remains a (open) problem.

Another family of methods called sample methods have also been used. One type uses moving particles on the surface, with repulsion forces which make it possible to spread the particles over the surface [10]. Another type starts from an initial set of sample points on the surface and refine it by inserting new points of the surface, in order to improve the approximation level. Techniques based on Delaunay triangulation of these points have been used for instance for this purpose [11,12].

In the presence of singularities, these methods are not producing correct output and refining the precision parameter of these algorithms increases the number of output points, without solving these singularity problems.

In a completely different direction, methods inspired by Cylindrical Algebraic Decomposition [13] have been proposed to analyse the topology of algebraic curves or surfaces, even in singular cases. The approach has been applied successfully to curves in 2D, 3D, 4D [2,14,15,16,17] and to surfaces [18,19,20]. They use projection techniques based on a conceptual sweeping line/plane perpendicular to some axis, and detect the critical topological events, such as tangents to the sweeping planes and singularities. They involve the exact computation of critical points and genericity condition tests or adjacency tests. The final output of these methods is a topological complex of points, segments, triangles isotopic to the curve or the surface.

They assume exact input equations and rely of the computation of subresultant sequences or calculus with algebraic numbers. This can be a bottleneck in many examples with large degree and large coefficients. Moreover, they are delicate to apply with approximate computation.

In order to combine approximation properties with certification and adaptivity, we consider subdivision methods, which proceed from a large input domain and subdivide it if a regularity criterion is not satisfied. This regularity criterion is designed so that the topology of the curve or the surface lying in the domain can be determined easily. Unfortunately, this type of approach has usually difficulties when singular points exists in the domain, which make the regularity test failing and the subdivision process going one until some threshold ϵ on the size of the boxes is reached. The obstacle comes from the fact that near a singularity, what ever scale of approximation you choose, the shape or topology of the algebraic objects remains similar. In this paper, we will focus on a regularity criterion which allows to deduce the topology of the variety in a domain, from its intersection with the boundary of the domain. We exploit the local conic structure near points on an algebraic curve or surface, to device algorithms which for a small enough threshold ϵ compute the correct topology, even in the presence of singular points.

These subdivision methods have been already used for solving several equations [21,22]. We recall the recent improvements proposed in [23], which rely on a polynomial solver as the basic ingredient of algorithms for curves and surfaces. Extension of this approach to higher dimensional objects have also been considered [24,14,25,26]. We will recall the subdivision method described in [27] for curves in 3D. It is based on a criterion, which allows us to detect easily when the topology of the curve in a box is uniquely determined from its intersection with the boundary of the box. The treatment of smooth surfaces by subdivision methods as been described in [28]. In this paper, we extend this approach by encompassing the singular case. The approach relies on the computation of the topology of a special curve on the surface, called the polar variety. It is used to detect points at which the surface has a conic structure, meaning we can tell what the topology is by uniquely looking at its intersection with the boundary of a box around the point.

Definitions

Before going into details, here are the notations and definitions we use hereafter:

- For subset domain $S \subset \mathbb{R}^n$, we denote by S° its interior, by $\overline{S}$ its closure, and by ∂S its boundary.
- We call any closed set D such that $\overline{D^\circ} = D$, a domain.
- We call any connected smooth curve $\mathcal{C}$ such that $\mathcal{C} \cap \partial D \neq \emptyset$ and $\mathcal{C} \cap D^\circ \neq \emptyset$, a branch (relative to a domain D), .
- We call any connected submanifold (possibly with boundary) included in the surface (resp. curve), with same the dimension as the surface (resp. curve),a patch of a surface (resp. curve).
- We call a point where the tangent space to the surface (resp. curve) contains the direction x (resp. y, z), a x-critical point (resp. y,z-critical point) of a surface (resp. curve).
- For any point $p \in \mathbb{R}^n$ and $r > 0$, the hypersphere (resp. disk) centered at p of radius r is denoted by $S(p,r) = \{q \in \mathbb{R}^n; \|q-p\| = r\}$ (resp. $D(p,r) = \{q \in \mathbb{R}^n; \|q-p\| \leq r\}$).
- By expressions such as 'topology computation' or 'determine the topology' we mean that we generate an embedded triangulation whose vertices are on the original surface (resp. curve) and which is homeomorphic to that surface

(resp. curve). Our construction actually leads to an embedded triangulation that is isotopic (meaning there is a continuous injective deformation of one onto the other) to the original variety, but this would require some more careful examination of our construction.

- For a box $B = [a_0, b_0] \times [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^3$, its x-faces (resp. y-face, z-face) are its faces normal to the direction x (resp. y, z).

  The size of B, denoted by $|B|$, is $|B| = \max\{|b_i - a_i|; i = 1, \ldots, n\}$.

2 Polynomial Equations

This section recalls the theoretical background of Bernstein polynomial representation and how it is related to the problem we want to solve.

2.1 Bernstein Basis Representation

Given an arbitrary univariate polynomial function $f(x) \in \mathbb{K}$, we can convert it to a representation of degree d in Bernstein basis, which is defined by:

$$f(x) = \sum_i b_i B_i^d(x), \text{ and} \tag{1}$$

$$B_i^d(x) = \binom{d}{i} x^i (1-x)^{d-i} \tag{2}$$

where b_i is usually referred as controlling coefficients. Such conversion is done through a basis conversion [29]. The above formula can be generalized to an arbitrary interval $[a, b]$ by a variable substitution $x' = (b-a)x + a$. We denote by $B_d^i(x; a, b) \binom{d}{i} (x-a)^i (b-x)^{d-i} (b-a)^{-d}$ the corresponding Bernstein basis on $[a, b]$. There are several useful properties regarding Bernstein basis given as follows:

- *Convex-Hull Properties*: As $\sum_i B_d^i(x; a, b) \equiv 1$ and $\forall x \in [a, b], B_d^i(x; a, b) \geq 0$ where $i = 0, ..., d$, the graph of $f(x) = 0$, which is given by $(x, f(x))$, should always lie within the convex-hull defined by the control coefficients $(\frac{i}{d}, b_i)$ [29].
- *Subdivision* (de Casteljau): Given $t_0 \in [0, 1]$, $f(x)$ can be represented by:

$$f(x) = \sum_{i=0}^{d} b_0^{(i)} B_d^i(x; a, c) = \sum_{i=0}^{d} b_i^{(d-i)} B_d^i(x; c, b), \text{ where} \tag{3}$$

$$b_i^{(k)} = (1-t_0) b_i^{(k-1)} + t_0 b_{i+1}^{(k-1)} \text{ and } c = (1-t_0)a + t_0 b. \tag{4}$$

By a direct extension to the multivariate case, any polynomial $f(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d_i in the variable x_i, can be decomposed as:

$$f(x_1, \ldots, x_n) = \sum_{i_1=0}^{d_1} \cdots \sum_{i_n=0}^{d_n} b_{i_1,\ldots,i_n} B_{d_1}^{i_1}(x_1; a_1, b_1) \cdots B_{d_n}^{i_n}(x_n; a_n, b_n).$$

where $(B_{d_1}^{i_1}(x_1; a_1, b_1) \cdots B_{d_n}^{i_n}(x_n; a_n, b_n))_{0 \le i_1 \le d_1, \ldots, 0 \le i_n \le d_n}$ is the tensor product Bernstein basis on the domain $B := [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ and $\mathbf{b} = (b_{i_1,\ldots,i_n})_{0 \le i_1 \le d_1, \ldots, 0 \le i_n \le d_n}$ are the control coefficients of f on B. The polynomial f is represented in this basis by the n^{th} order tensor of control coefficients $\mathbf{b} = (b_{i_1,\ldots,i_n})_{0 \le i \le d_1, 0 \le j \le d_2, 0 \le k \le d_3}$.

De Casteljau algorithm also applies in each of the direction x_i, , $i = 1, \ldots, n$ so that we can split this representation in these directions. We use the following properties to isolate the roots:

This representation provides a simple way to tell the sign of a function in a domain B.

Lemma 2.1. *If all the coefficients $b_{i_1,\ldots,i_n}$ of f in Bernstein basis of $B := [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ have the same sign $\epsilon \in \{-1, 1\}$, then $\epsilon f(\mathbf{x}) > 0$ for $\mathbf{x} \in B$.*

Proof. As the Bernstein basis elements of the domain B are positive on B and their sum is 1, for $\mathbf{x} \in B$, $f(\mathbf{x})$ is a barycentric combination of the coefficients $b_{i_1,\ldots,i_n}$, of sign ϵ. This $f(\mathbf{x})$ is of sign ϵ.

A consequence is the following interesting property:

Lemma 2.2. *Let f and g by polynomials of degree d_i in x_i $(i = 1, \ldots, n)$ and let $b_{i_1,\ldots,i_n}$ and $c_{i_1,\ldots,i_n}$ be their coefficients in the Bernstein basis of $B := [a_1, b_1] \times \cdots \times [a_n, b_n]$. If $b_{i_1,\ldots,i_n} \le c_{i_1,\ldots,i_n}$ for $0 \le i_j \le d_j, j = 1, \ldots, n$ then $f(\mathbf{x}) \le g(\mathbf{x})$ for $\mathbf{x} \in B$.*

It will be used in algorithm for computing the topology of implicit curves and surfaces as follows. When the input coefficients of a polynomial f are large rational numbers, instead of working with this expensive arithmetic, we will first compute its coefficients in the Bernstein basis of the given domain B, then normalize them and finally round them up and down to machine precision arithmetic (ie. double). This produces two enveloping functions $\underline{f}, \bar{f}$ with the property:

$$\underline{f}(\mathbf{x}) \le f(\mathbf{x}) \le \bar{f}(\mathbf{x}), \forall \mathbf{x} \in B.$$

These two enveloping polynomials can be used to test sign conditions and regularity criteria, providing certificated results in many situations.

2.2 Univariate Subdivision Solver

Another interesting property of this representation related to Descartes rule of signs is that there is a simple and yet efficient test for the existence of real roots in a given interval. It is based on the number of sign variation $V(\mathbf{b})$ of the sequence $\mathbf{b} = [b_1, \ldots, b_k]$ that we define recursively as follows:

$$V(\mathbf{b}_{k+1}) = V(\mathbf{b}_k) + \begin{cases} 1, \text{if } b_i b_{i+1} < 0 \\ 0, \text{else} \end{cases} \tag{5}$$

With this definition, we have:

Proposition 2.1. *Given a polynomial* $f(x) = \sum_i^n b_i B_i^d(x; a, b)$*, the number* N *of real roots of* f *on* $]a, b[$ *is less than or equal to* $V(\boldsymbol{b})$*, where* $\boldsymbol{b} = (b_i), i = 1, ..., n$ *and* $N \equiv V(\boldsymbol{b}) \mod 2$.

With this proposition,

- if $V(\mathbf{b}) = 0$, the number of real roots of f in $[a, b]$ is 0;
- if $V(\mathbf{b}) = 1$, the number of real roots of f in $[a, b]$ is 1.

This yields the following simple and efficient algorithm [30]:

Algorithm 1
INPUT: *A precision* ϵ *and a polynomial* f *represented in the Bernstein basis of an interval* $[b, a]$*:* $f = (\boldsymbol{b}, [a, b])$.

- *Compute the number of sign changes* $V(\boldsymbol{b})$.
- *If* $V(\boldsymbol{b}) > 1$ *and* $|b - a| > \epsilon$*, subdivide the representation into two sub-representations* $\boldsymbol{b}^-, \boldsymbol{b}^+$*, corresponding to the two halves of the input interval and apply recursively the algorithm to them.*
- *If* $V(\boldsymbol{b}) > 1$ *and* $|b - a| < \epsilon$*, output the* $\epsilon/2$*-root* $(a + b)/2$ *with multiplicity* $V(\boldsymbol{b})$.
- *If* $V(\boldsymbol{b}) = 0$*, remove the interval* $[a, b]$.
- *If* $V(\boldsymbol{b}) = 1$*, the interval contains one root, that can be isolated with precision* ϵ.

OUTPUT: *list of subintervals of* $[a, b]$ *containing exactly one real root of* f *or of* ϵ*-roots with their multiplicities.*

In the presence of a multiple root, the number of sign changes of a representation containing a multiple root is bigger than 2, and the algorithm splits the box until its size is smaller than ϵ.

To analyze the behavior of the algorithm, a partial inverse of Descartes' rule and lower bounds on the distance between roots of a polynomial have been used. It is proved that the complexity of isolating the roots of a polynomial of degree d, with integer coefficients of bit size $\leq \tau$ is bounded by $\mathcal{O}(d^4\tau^2)$ up to some polylog factors. See [31,30] for more details.

Notice that this localization algorithm extends naturally to B-splines, which are piecewise polynomial functions [29].

The approach can also be extended to polynomials with interval coefficients, by counting 1 sign variation for a sign sub-sequence $+, ?, -$ or $-, ?, +$; 2 sign variations for a sign sub-sequence $+, ?, +$ or $-, ?, -$; 1 sign variation for a sign sub-sequence $?, ?$, where ? is the sign of an interval containing 0. Again in this case, if a family $\overline{f}$ of polynomials is represented by the sequence of intervals $\bar{\mathbf{b}} = [\bar{b}_0, \ldots, \bar{b}_d]$ in the Bernstein basis of the interval $[a, b]$

- if $V(\bar{\mathbf{b}}) = 1$, all the polynomials of the family $\overline{f}$ have one root in $[a, b]$,
- if $V(\bar{\mathbf{b}}) = 0$, all the polynomials of the family $\overline{f}$ have no roots in $[a, b]$.

The same subdivision algorithm can be applied to polynomials with interval coefficients, using interval arithmetic. This yields either intervals of size smaller than ϵ, which might contain the roots of $f = 0$ in $[a, b]$ or isolating intervals for all the polynomials of the family defined by the interval coefficients.