Partial Differential Equations

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Lawrence C. Evans

Graduate Studies in Mathematics

Volume 19

American Mathematical Society
Providence, Rhode Island

1991 *Mathematics Subject Classification.* Primary 35-xx; Secondary 49-xx, 47Hxx.

ABSTRACT. This text surveys a wide variety of topics in the mathematical theory of partial differential equations (PDE). The primary topics are: representation formulas for solutions, theory for linear PDE, theory for nonlinear PDE.

Library of Congress Cataloging-in-Publication Data

Evans, Lawrence C., 1949–
Partial differential equations / Lawrence C. Evans.
p. cm. - (Graduate studies in mathematics, ISSN 1065-7339 ; v. 19)
Includes bibliographical references and index.
ISBN 0-8218-0772-2 (alk. paper)
1. Differential equations, Partial. I. Title. II. Series.
QA377.E95 1998 97-41033
515'.353-dc21 CIP

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10 9 8 7 6 5 4 3 2 1 03 02 01 00 99 98

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I dedicate this book to the memory of my parents,

LAWRENCE S. EVANS and LOUISE J. EVANS.

PREFACE

I present in this book a wide-ranging survey of many important topics in the theory of partial differential equations (PDE), with particular emphasis on various modern approaches. I have made a huge number of editorial decisions about what to keep and what to toss out, and can only claim that this selection seems to me about right. I of course include the usual formulas for solutions of the usual linear PDE, but also devote large amounts of exposition to energy methods within Sobolev spaces, to the calculus of variations, to conservation laws, etc.

My general working principles in the writing have been these:

a. PDE theory is (mostly) not restricted to two independent variables. Many texts describe PDE as if functions of the two variables (x, y) or (x, t) were all that matter. This emphasis seems to me misleading, as modern discoveries concerning many types of equations, both linear and nonlinear, have allowed for the rigorous treatment of these in any number of dimensions. I also find it unsatisfactory to "classify" partial differential equations: this is possible in two variables, but creates the false impression that there is some kind of general and useful classification scheme available in general.

b. Many interesting equations are nonlinear. My view is that overall we know too much about linear PDE and too little about nonlinear PDE. I have accordingly introduced nonlinear concepts early in the text and have tried hard to emphasize everywhere nonlinear analogues of the linear theory.

c. Understanding generalized solutions is fundamental. Many of the partial differential equations we study, especially nonlinear first-order equations, do not in general possess smooth solutions. It is therefore essential to devise some kind of proper notion of generalized or weak solution. This is an important but subtle undertaking, and much of the hardest material in this book concerns the uniqueness of appropriately defined weak solutions.

d. PDE theory is not a branch of functional analysis. Whereas certain classes of equations can profitably be viewed as generating abstract operators between Banach spaces, the insistence on an overly abstract viewpoint, and consequent ignoring of deep calculus and measure theoretic estimates, is ultimately limiting.

e. Notation is a nightmare. I have really tried to introduce consistent notation, which works for all the important classes of equations studied. This attempt is sometimes at variance with notational conventions within a subarea.

f. Good theory is (almost) as useful as exact formulas. I incorporate this principle into the overall organization of the text, which is subdivided into three parts, roughly mimicking the historical development of PDE theory itself. Part I concerns the search for explicit formulas for solutions, and Part II the abandoning of this quest in favor of general theory asserting the existence and other properties of solutions for linear equations. Part III is the mostly modern endeavor of fashioning general theory for important classes of nonlinear PDE.

Let me also explicitly comment here that I intend the development within each section to be rigorous and complete (exceptions being the frankly heuristic treatment of asymptotics in §4.5 and an occasional reference to a research paper). This means that even locally within each chapter the topics do not necessarily progress logically from "easy" to "hard" concepts. There are many difficult proofs and computations early on, but as compensation many easier ideas later. The student should certainly omit on first reading some of the more arcane proofs.

I wish next to emphasize that this is a *textbook*, and not a reference book. I have tried everywhere to present the essential ideas in the clearest possible settings, and therefore have almost never established sharp versions of any of the theorems. Research articles and advanced monographs, many of them listed in the Bibliography, provide such precision and generality. My goal has rather been to explain, as best I can, the many fundamental ideas of the subject within fairly simple contexts.

I have greatly profited from the comments and thoughtful suggestions of many of my colleagues, friends and students, in particular: S. Antman, J. Bang, X. Chen, A. Chorin, M. Christ, J. Cima, P. Colella, J. Cooper,

M. Crandall, B. Driver, M. Feldman, M. Fitzpatrick, R. Gariepy, J. Goldstein, D. Gomes, O. Hald, W. Han, W. Hrusa, T. Ilmanen, I. Ishii, I. Israel, R. Jerrard, C. Jones, B. Kawohl, S. Koike, J. Lewis, T.-P. Liu, H. Lopes, J. McLaughlin, K. Miller, J. Morford, J. Neu, M. Portilheiro, J. Ralston, F. Rezakhanlou, W. Schlag, D. Serre, P. Souganidis, J. Strain, W. Strauss, M. Struwe, R. Temam, B. Tvedt, J.-L. Vazquez, M. Weinstein, P. Wolfe, and Y. Zheng.

I especially thank Tai-Ping Liu for many years ago writing out for me the first draft of what is now Chapter 11.

I am extremely grateful for the suggestions and lists of mistakes from earlier drafts of this book sent to me by many readers, and I encourage others to send me their comments, at evans@math.berkeley.edu. I have come to realize that I must be more than slightly mad to try to write a book of this length and complexity, but I am not yet crazy enough to think that I have made no mistakes. **I will therefore maintain a listing of errors which come to light, and will make this accessible through the math.berkeley.edu homepage.**

Faye Yeager at UC Berkeley has done a really magnificent job typing and updating these notes, and Jaya Nagendra heroically typed an earlier version at the University of Maryland. My deepest thanks to both.

I have been supported by the NSF during much of the writing, most recently under grant DMS-9424342.

LCE

August, 1997

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INTRODUCTION

This chapter surveys the principal theoretical issues concerning the solving of partial differential equations.

To follow the subsequent discussion, the reader should first of all turn to Appendix A and look over the notation presented there, particularly the multiindex notation for partial derivatives.

1.1. PARTIAL DIFFERENTIAL EQUATIONS

A *partial differential equation (PDE)* is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Using the notation explained in Appendix A, we can write out symbolically a typical PDE, as follows. Fix an integer $k \geq 1$ and let U denote an open subset of $\mathbb{R}^n$.

DEFINITION. *An expression of the form*

$$(1) \qquad F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0 \quad (x \in U)$$

is called a k$^{\text{th}}$-order partial differential equation, *where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$