FUNDAMENTAL SCIENCES *Mathematics*

GROUP REPRESENTATION THEORY

Meinolf Geck, Donna Testerman
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Preface

This volume gives a general view of the main activities which took place during the research semester "Group Representation Theory", held in Lausanne, Switzerland, from January to June, 2005. This program included five graduate courses, two workshops, one conference, and numerous seminars. It was hosted by the Bernoulli Centre of the École Polytechnique Fédérale de Lausanne (EPFL) and was funded jointly by EPFL and the Swiss National Foundation.

The volume consists of a collection of independent contributions. There is no aim at uniformity and the diversity of the styles reflects the individuality of the authors. The level of exposition is intended for graduate students and researchers in representation theory.

The first part of the semester was concerned with the interplay between the representation theory of finite groups, cohomology, and topology. A one-week workshop "Topology, representation theory, cohomology" was held in April and its main purpose was to gather topologists and representation theorists working on fusion systems and p-local finite groups. Two introductory papers are published here on some of the algebraic aspects of the theory (a survey on the topological aspects appears elsewhere). The first is an introduction to fusion systems by M. Linckelmann and the second is a survey by R. Kessar of the important case of blocks. Finally many developments in this area include representations of categories and related cohomological methods, which are presented here by P. J. Webb.

In this first part of the semester, J. F. Carlson gave a graduate course entitled "Cohomology and representations of finite groups", which is published here. In addition, a small workshop on the recent classification of endo-permutation modules for p-groups was held in April and a survey by J. Thévenaz on this classification is included in this volume.

The second part of the semester was dedicated to algebraic groups and finite reductive groups. The research areas of the participants covered wide-ranging topics in the representation theory of algebraic and finite reductive groups and Hecke algebras, as well as the interaction between the representation theory and subgroup structure of semisimple algebraic groups. The program culminated in a one-week conference "Algebraic groups and reductive groups" held in June. This volume contains an introduction to representations of algebraic groups, which was presented by S. Donkin during a small workshop on algebraic groups in May. This workshop paved the way for several graduate courses:

"Representations of Hecke algebras" by M. Geck, "Topics in algebraic groups" by G. Seitz, "Finite reductive groups and spetses" by M. Broué, and last but not least "On finite subgroups of Lie groups" by J.-P. Serre.

For all of these graduate courses, expanded notes were prepared by the authors and are published here.

It is a pleasure to thank the Bernoulli Center and its staff for their dedicated work during the program. We wish to thank as well all the participants who contributed to the success of the semester and in particular the speakers who agreed to write articles for this volume.

Meinolf Geck, Donna Testerman, and Jacques Thévenaz

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Part I

Representations, Functors and Cohomology

Cohomology and Representation Theory

JON F. CARLSON[(1)]

1. Introduction

In these lectures we shall consider a few aspects of the interaction between group cohomology and group representation theory. That interaction has grown tremendously in the last thirty years to the point that homological methods are now standard in modular representation theory. The subject is much too large to give a complete picture in the space of a half semester of lectures. Consequently, we will concentrate on the methods and results required for one application: the classification of endotrivial modules. The classification is a statement about modules over group algebras and makes almost no mention of homological algebra or cohomology. Yet its proof relies in fundamental ways on the theory of support varieties, on the computations of the cohomology rings of extraspecial groups and on several other items from group cohomology.

The endotrivial modules were introduced by Dade in [22], who showed them to be the building blocks for the endopermutation modules. The endopermutation modules are the sources for the simple module for p-solvable groups [28] and are also of interest in block theory and categorical equivalences [14]. Dade also proved a classification for the endotrivial modules for an abelian p-group. Puig [29] showed that the group of endotrivial modules is finitely generated. Following those beginnings there was a long period with no big progress. Then in the middle 1990's, Alperin [1] found the torsion free rank of the group of endotrivial modules. Alperin's result was proved independently by Bouc and Thévenaz [10]. A few years later, Thévenaz and the author [15, 16, 17] characterized the torsion part of the group and showed that Alperin's generators, indeed, generated the torsion free part. Another collection of generators was

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constructed in [13]. Very recently, Bouc [9] has used the classification of endotrivial modules to complete a characterization of the Dade group of endopermutation modules of a p-group. There has also been some progress made on classifying the endotrivial modules for groups which are not p-groups [19].

A key point in the proof of the classification of endotrivial modules was the development of an effective method of computing an upper bound for the dimensions of endotrivial modules based on group cohomology. This method relies on an explicit proof of Quillen's Dimension Theorem in terms of the vanishing of certain cohomology products. The proof first appeared in [12] and we present some part of it in Section 5.

In the final section of the notes we present a proof of one piece of the classification of the endotrivial modules. The theorem establishes the rank of the torsion free part of the group of endotrivial modules. We use an entirely different method from the proof given by Alperin [1]. Our proof is much more representative of the type of techniques that were used in [16] and [17]. Some other parts of the classification are sketched in other sections. For example, near the end of Section 3 we show how to construct exotic endotrivial modules for the quaternion group. This particular construction turned out to be a key to the entire proof of the classification.

We have augmented the text of the notes with some exercises. Some of the exercises are fairly difficult. Perhaps in a few cases, these should be considered more to be pondered than to be solved.

In general we assume a basic knowledge of homological algebra and group representations. The first two or three sections will cover foundational material and be treated mostly as review. In the later sections we encounter some theorems whose proofs, because of time constraints, will be omitted or only sketched.

Throughout these notes, the symbol k denotes a field of prime characteristic p. In general, we assume that k is algebraically closed, though for many of the theorems, this restriction is not necessary. All modules are left unital modules unless stated otherwise. The tensor product $\otimes$ means $\otimes_k$. The k-dual of a kG-module or k-vector space M is denoted M^*. All modules will be assumed to be finitely generated. Recall that modules over a finite dimensional algebra satisfy the Krull-Schmidt Theorem. That is, every (finitely generated) module can be written uniquely (up to isomorphism and order of the factors) as a direct sum of indecomposable modules.

For most of the basic material, no references are given. The results can be found in one or all of the basic text books on the subject [6, 18, 24].

Acknowlegments: The author would like to thank the Centre Bernoulli and the École Polytechnique Fédérale de Lausanne for support and general help during the period of these lectures.

2. Modules over p-groups

In this section we explore the group algebras of p-groups and their representations. All of the material in this section is standard and can be found in almost any text that deals with modular representation theory. Some of the results of the section hold for all finite groups and not just p-groups. The reader who is unfamiliar with some of the material in this section is encouraged to work through the exercises in some detail.

Throughout this section assume that G is a finite group. We specialize to p-groups later in the section. First we need some basics on group algebras.

Hopf algebras, tensor products and duals

We have a Hopf algebra structure $kG \longrightarrow kG \times kG$ given on basis elements by $g \mapsto (g, g)$. This means that if M and N are kG-modules, then so is $M \otimes N$ with the action of $g \in G$ defined by $g(m \otimes n) = gm \otimes gn$ for $m \in M$ and $n \in N$. Likewise we make $\mathrm{Hom}_k(M, N)$ into a kG-module by letting $(gf)(m) = gf(g^{-1}m)$ for all $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$.

Exercise 2.1. Prove that $M^* \otimes N \cong \mathrm{Hom}_k(M, N)$ by the map which sends $\lambda \otimes n$ to f where $f(m) = \lambda(m)n$ for all $\lambda \in M^*$, $n \in N$ and $m \in M$. Show also that the isomorphism is natural in both variables.

Exercise 2.2. Suppose that $G = \langle x, y \rangle$ is an elementary abelian group of order 4, and k has characteristic 2. Let $M = M_\alpha$ be the kG-module of dimension 2 for which the actions of x and y are given by the matrices

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

for some element $\alpha \in k$ and some basis $\{m_1, m_2\}$ of M. Find a decomposition of $M \otimes M$ into a direct sum of indecomposable modules. Do the same for $M_\alpha \otimes M_\beta$ where α and β are different elements of k.

Symmetric and self-injective algebras

Let $\sigma : kG \longrightarrow k$ be the k-vector space homomorphism defined by $\sigma(\sum a_g \cdot g) = a_1$. That is, σ applied to an element of kG returns the coefficient on the identity element of G. Define a nondegenerate symmetric bilinear form $(\ ,\ ) : kG \times kG \longrightarrow k$ by the rule $(\alpha, \beta) = \sigma(\alpha \cdot \beta)$. Nondegenerate means that if $(\alpha, \beta) = 0$ for all β or if $(\beta, \alpha) = 0$ for all β, then $\alpha = 0$. It can be seen that the form is G-invariant in the sense that $(\alpha g, \beta) = (\alpha, g\beta)$ for all $g \in G$, $\alpha, \beta \in kG$. The form proves that kG is a symmetric algebra. That is, there is an isomorphism $\phi : kG \cong kG^*$ given by $\phi(\alpha) = (\alpha,\ )$. A consequence of this is the following:

Theorem 2.3. *The group algebra kG is a self-injective algebra. That is, every finitely generated projective module is injective, and conversely, every finitely generated injective module is projective.*

Exercise 2.4. Prove the theorem. Show first that finitely generated free modules are injective using the duality.

Module categories

We let $\mathbf{mod}(kG)$ denote the category of finitely generated kG-modules. Let $\mathbf{stmod}(kG)$ denote the stable category of kG-modules modulo projectives. The objects in $\mathbf{stmod}(kG)$ are the same as those in $\mathbf{mod}(kG)$, but the morphisms from modules M to N are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) \;=\; \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M,N)$ is the set of all homomorphisms from M to N that factor through a projective module. We say that $\alpha : M \longrightarrow N$ factors through a projective module if there exist a projective module P and maps $\mu : M \longrightarrow P$ and $\nu : P \longrightarrow N$ such that $\nu\mu = \alpha$.

Induction and Frobenius reciprocity

Suppose that H is a subgroup of G. If M is a kG-module, we let M_H denote the restriction of M to a kH-module. If some emphasis is required we use the symbol $M_{\downarrow H}$ to denote the restriction. If N is a kH-module, then the induced module $N^{\uparrow G} = kG \otimes_{kH} N$ is a kG-module with the action of G on the left. Both restriction and induction are functors on the module categories. There are two results relating induction and restriction that are very useful to us. The first is known as Frobenius Reciprocity.

Theorem 2.5. *Let M be a kG-module and N a kH-module. Then*

$$M \otimes N^{\uparrow G} \cong (M_H \otimes N)^{\uparrow G}.$$

The isomorphism is given by the map $m \otimes (g \otimes n) \mapsto g \otimes (g^{-1}m \otimes n)$ for all $g \in G$, $m \in M$ and $n \in N$. In the other direction, the map sends $g \otimes (m \otimes n) \mapsto gm \otimes (g \otimes n)$.

The other result is known as the Mackey formula.

Theorem 2.6. *Suppose that M is a finitely generated kH-module for H a subgroup of G. Let K be another subgroup of G. Then*

$$(M^{\uparrow G})_{\downarrow K} \cong \sum_{KxH} ((x \otimes M)_{K \cap xHx^{-1}})^{\uparrow K} = \sum_{KxH} (x \otimes M_{x^{-1}Kx \cap H})^{\uparrow K},$$

where the sum is indexed by the $K-H$-double cosets in G.

Now notice that if $H = \{1\}$, the identity subgroup, and if k_H is the trivial kH-module, then $k_H^{\uparrow G} \cong kG$ as left kG-modules. A consequence of this and Frobenius Reciprocity is the following.

Exercise 2.7. Suppose that P is a projective kG-module and that M is any kG-module. Prove that $M \otimes P$ is projective.

Degree shifting

The notation and ideas of this section are vital for the rest of the course. First we recall Schanuel's Lemma.

Proposition 2.8. *Let R be a ring and let M be an R-module. Suppose that P_1 and P_2 are projective modules and that $\theta_1 : P_1 \longrightarrow M$, $\theta_2 : P_2 \longrightarrow M$ are surjective homomorphisms. Let K_i be the kernel of θ_i for $i = 1, 2$. Then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.*

If M is a finitely generated kG-module, then there exists a finitely generated projective cover $\theta : P \longrightarrow M$. That is, P is a projective module of least dimension such that there is a surjective homomorphism (theta) onto M. We denote the kernel of θ by $\Omega(M)$. Notice that $\Omega(M)$ has no projective submodule, because if Q were a projective submodule of $\Omega(M)$, then Q would be a projective and also injective submodule of P. Hence, Q would be a direct summand of P, thus contradicting the minimality of P. Moreover, $\Omega(M)$ is uniquely defined in the sense that if $\gamma : Q \longrightarrow M$ is any surjective homomorphism with Q projective, then by Schanuel's Lemma the kernel of γ is $\Omega(M) \oplus (\text{proj})$, where by $\oplus\,(\text{proj})$ we mean the direct sum with some projective module.

Inductively, we define, $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ for all natural numbers $n > 1$. For the reasons given, $\Omega^n(M)$ is well defined up to isomorphism. The module M has an injective hull given by $\theta : M \longrightarrow Q$ where Q is a smallest injective (projective) module into which M injects. Then the cokernel of θ is denoted $\Omega^{-1}(M)$ and has no injective (hence projective) submodules. Iterating, we define $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for $n > 0$. We let $\Omega^0(M)$ be the nonprojective part of M, the direct sum of all of the nonprojective indecomposable summands of M.

With the above definitions and some facts that we know about projective modules, we can prove the following very useful result.

Exercise 2.9. Suppose that M and N are kG-modules and m and n are any integers. Then

(i) $\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{m+n}(M \otimes N) \oplus (\text{proj})$, and

(ii) $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$.

Definition 2.10. A kG-module is an endotrivial module if its k-endomorphism ring is the direct sum of a trivial module and a projective module. That is, M is endotrivial if and only if

$$\text{Hom}_k(M, M) \cong M^* \otimes M \cong k \oplus (\text{proj}).$$

The previous exercise shows that for any integer n, $\Omega^n(k)$ is an endotrivial module.

Group algebras of p-groups

Suppose now that G is a p-group. Note that if $x \in G$, then $(x-1)^{p^n} = x^{p^n} - 1 = 0$, provided p^n is the order of x. Consequently, the augmentation ideal $I(kG)$ of kG, the ideal generated by all $x - 1$ for x in G, is generated by nilpotent elements. Slightly harder to prove is the following.