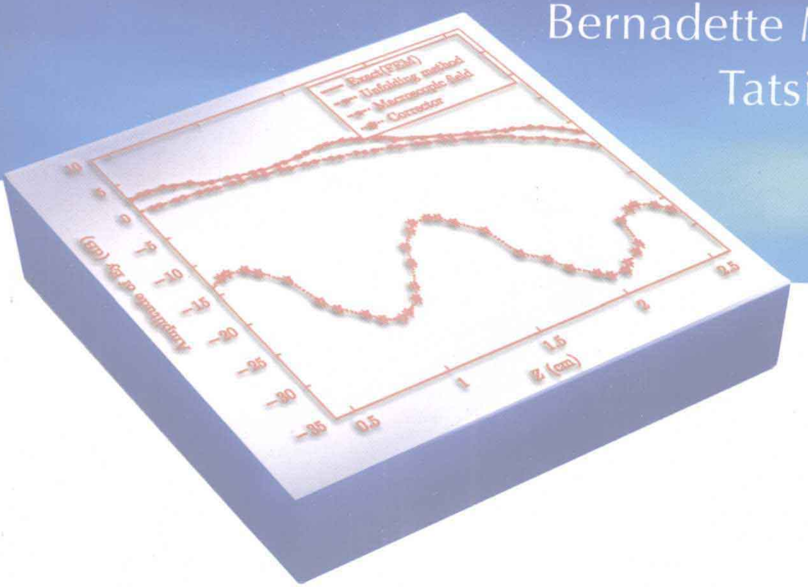
Series in Contemporary Applied Mathematics
CAM 16

Multiscale Problems

Theory, Numerical Approximation and Applications

多尺度问题——理论、数值逼近及应用

Alain Damlamian
Bernadette Miara
Tatsien Li
editors



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Duochidu Wenti

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Preface

The ISFMA Symposium "Multiscale Problems: Theory, Numerical Approximation and Applications" was held in May, 4–16, 2009 at Fudan University in Shanghai. Its aim was to introduce graduate students and post-doctors to the newest developments related to the analysis of problems in which several scales are presented.

This volume gathers the notes corresponding to the lectures given by the following professors: Alain Damlamian (Laboratoire d'Analyse et Mathématiques Appliquées, Université Paris-Est, Paris12-Val de Marne, France), Gabriel Nguetseng (Department of Mathematics, University of Yaounde 1, Cameroon), Georges Griso (Laboratoire d'Analyse Numérique J.L. Lions, Université P. et M. Curie, Paris, France), Patrizia Donato (Université de Rouen, St Etienne du Rouvray, France), Dominique Blanchard (Université de Rouen, Saint Etienne du Rouvray, France), Bernadette Miara (Département de Modélisation et Simulation Numérique, Université Paris-Est, Ecole Supérieure d'Ingénieurs en Electronique et Electrotechnique, Noisy-le-Grand, France) and Assyr Abdulle (Section of Mathematics, Swiss Federal Institute of Technology, Switzerland).

The contributions listed below cover a wide range of topics in theory, numerical approximation with finite elements and applications in the fields of elasticity and fluid mechanics.

Chapter 1, by Alain Damlamian, presents a general introduction to the theory of homogenization in the periodic case.

Chapter 2, by Alain Damlamian, is on the periodic unfolding method, a very efficient and recent method for periodic homogenization.

Chapter 3, by Gabriel Nguetseng and Lazarus Signing, presents the homogenization of the stationary Navier-Stokes equations in fixed or variable domains occupied by porous media.

Chapter 4, by Patricia Donato, concerns the homogenization of a class of imperfect transmission problems.

Chapters 5 and 6, by Georges Griso, present a new approach to the decomposition of displacements in the case of thin structures (chapter 5) and rods with applications to the asymptotic behavior of nonlinear elastic rods (chapter 6).

Chapter 7, by Dominique Blanchard, concerns the elastic behavior for the junction of a periodic family of rods with a plate as the limit of a 3D elastic body.

Chapter 8, by Bernadette Miara, presents the theory and numerical simulation for the multi-scale modelling of new composites.

Chapter 9, by Assyr Abdulle, is a presentation and analysis of heterogeneous multiscale finite element methods (HMFEM); the a priori and a posteriori analysis of such numerical methods is investigated and a general framework to perform such analysis is given.

We are very grateful to our colleagues who carefully prepared the paper version of their talks. It is also our pleasure to thank Ms. Zhou Chunlian whose kindness and professionalism guarantied the success of this Symposium.

We thank the Mathematical Center of Ministry of Education of China, the National Natural Science Foundation of China, the School of Mathematical Sciences, Fudan University, Shanghai Key Laboratory for Contemporary Applied Mathematics, the Research Center of Scientific Computing and Engineering and the Nonlinear Mathematical Modeling and Methods Laboratory, which sponsored the Symposium.

Finally, the editors would like to express their gratitude to Fudan University and the "Institut Sino-Français de Mathématiques Appliquées" (ISFMA) for their help and support.

Alain Damlamian, Bernadette Miara, Tatsien Li

January 2011

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An Introduction to Periodic Homogenization

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Abstract

The purpose of this series of lectures is to give a short introduction to the theory of Homogenization in the case of periodic problems. The main questions addressed by periodic homogenization are presented and the three "classical" tools are briefly explained. They serve as a general introduction for the various lectures given during the ISFMA Symposium "Multiscale Problems: Theory, Numerical Approximation and Applications".

1 Introduction

The mathematical theory of the homogenization was introduced in the late 1970's in order to describe the behaviour of composite materials and reticulated structures.

Composite materials have been used for a long time (for example, concrete is a composite material), and already back in the 1930's, their equivalent coefficients were studied, see for example [4]). But starting in the 1970's, new types of composites were introduced, and their manufacture became easier and more common. They were used for advanced technologies, and the design of these new composites became itself a high tech process. Today, they are used more and more in industry due to the enhanced properties they exhibit when properly manufactured. This phenomenon was examplified here in Shanghai at "The 6th China (Shanghai) Glass Fiber Composite Material Expo 2009", which took place at the Shanghai Everbright Convention & Exhibition Center on May 13, 14 and 15, 2009, during the time this ISFMA Symposium was held (see http://www.fiberglassexpo.cn).

Composite materials are characterized by the fact that they contain several finely mixed constituents in a structured way. They are designed to have a "better" behaviour than the average behaviour of its constituents. Well-known examples are the superconducting multifilamentary composites which are used in the composition of optical fibers, or the composite used in the aviation industry.

Generally speaking, the heterogeneities in a composite are small compared to the global dimensions. So, several scales are needed to describe such a material, one macroscopic scale, describing the global behaviour of the composite, and at least one microscopic scales describing the heterogeneities in the material. From the macroscopic point of view, the composite looks like a "homogeneous" material. The aim of "Homogenization Theory" is to give the precise macroscopic properties of the composite by taking into account the properties of the microscopic structure.

The classical model case, which we will consider throughout these lectures is the problem of the steady heat conduction in an isotropic composite. It has the advantages of being simple to explain, of being in scalar form, while presenting all the main complexities of the theory.

Consider first a homogeneous body occupying a physical domain Ω with thermal conductivity γ. For simplicity, assume that the material is isotropic, which means that γ is a scalar. Suppose that f represents the heat source and g the temperature on the surface $\partial\Omega$ of the body, which, for simplicity, we can assume to be equal to zero.

Then the temperature $u = u(x)$ at the point $x \in \Omega$ satisfies the following homogeneous Dirichlet problem

$$\begin{cases} -\operatorname{div}(\gamma \nabla u(x)) = f(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ∇u denotes the gradient of u defined by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right).$$

Since γ is constant, this can be rewritten in the form

$$\begin{cases} -\gamma \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta u = \operatorname{div}(\text{grad } u)$. The flux of the temperature is defined by

$$q = \gamma \text{ grad } u. \tag{1.3}$$

This is a classical elliptic boundary value problem and it is well-known that if f is sufficiently smooth, it admits a unique solution u which is twice differentiable and solves system (1.2) at every point x of Ω.

If now we consider a heterogeneous material occupying Ω, the thermal conductivity takes different values in each component of the composite. Hence, γ is now a function, which is discontinuous in Ω, since it jumps over surfaces which separate the constituents. To simplify, suppose we are in presence of a mixture of two materials, one occupying the subdomain Ω_1 and the second one the subdomain Ω_2, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2 \cup (\partial\Omega_1 \cap \partial\Omega_2)$.

Suppose also that the thermal conductivity of the body occupying Ω_1 is γ_1 and that of the body occupying Ω_2 is γ_2, i.e.

$$\gamma(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1, \\ \gamma_2 & \text{if } x \in \Omega_2. \end{cases}$$

Then the temperature and flux of the temperature in a point $x \in \Omega$ of the composite take respectively, the values

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1, \\ u_2(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$q = \begin{cases} q_1 = \gamma_1 \operatorname{grad} u_1 & \text{in } \Omega_1, \\ q_2 = \gamma_2 \operatorname{grad} u_2 & \text{in } \Omega_2. \end{cases}$$

The usual physical assumptions are the continuity of the temperature u and of the flux q at the interface of the two materials, i.e.

$$\begin{cases} u_1 = u_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \\ q_1.n_1 = q_2.n_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{1.4}$$

where n_i is the outward normal unit vector to $\partial\Omega_i$, $i = 1, 2$ and $n_1 = -n_2$ on $\partial\Omega_1 \cap \partial\Omega_2$. Therefore, the temperature u is the solution of the stationary thermal problem. The corresponding system (1.1) reads

$$\begin{cases} -\operatorname{div}(\gamma(x) \operatorname{grad} u(x)) = f(x) & \text{in } \Omega_1 \cup \Omega_2, \\ u = 0 \quad \text{on } \partial\Omega, \\ u_1 = u_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2, \\ q_1.n_1 = q_2.n_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{cases} \tag{1.5}$$

Formally, we can write this system in the form

$$\begin{cases} -\operatorname{div}(\gamma(x) \operatorname{grad} u(x)) = f(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Actually, the proper formulation of problem (1.6) (or (1.5)) is a variational equality, namely

$$\begin{cases} \text{find } u \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v dx, \quad \forall v \in H, \end{cases} \tag{1.7}$$

where H is the appropriate Sobolev space taking into account the boundary conditions on u (here $H = H_0^1(\Omega)$). In (1.7) the derivatives are taken in the sense of distributions, and (1.6) is equivalent to (1.7) when considered in the sense of distributions provided u is in the space H. In general, the sense to be given to (1.6) is only that u solves (1.7). The equation in (1.7) is checked for every v belonging to the space H, so v is usually called a test function.

Let us turn back to the question of the macroscopic behaviour of the composite material occupying Ω. Suppose that the heterogeneities are very small with respect to the size of Ω and that they are evenly distributed. This is a realistic assumption for a large class of applications. From the mathematical point of view, one can modelize this distribution by supposing that it is periodic (see Figure 1).

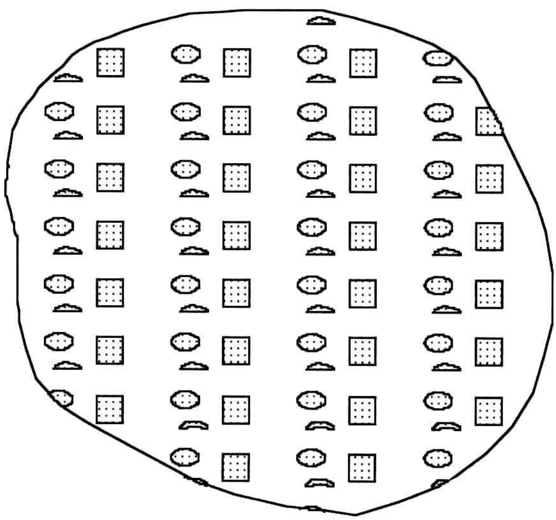

Figure 1

The scale of the periodicity (the size of the period) is small with respect to the macroscopic size in the problem. Let us call it ε.

The usual presentation of problem (1.7) is to consider coefficients γ

which are ε-periodic. Thus it reads

$$\begin{cases} \text{Find } u^\varepsilon \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v dx, \quad \forall v \in H. \end{cases} \tag{1.8}$$

For the purely periodic case, γ^ε in (1.8) has the form

$$\gamma^\varepsilon(x) = \gamma\Big(\frac{x}{\varepsilon}\Big) \quad \text{a.e. on } \mathbb{R}^N, \tag{1.9}$$

where γ is a given periodic function of period Y. This means that we are given a reference period Y, in which the reference heterogeneities are given. By definition (1.9), the heterogeneities in Ω are of size of order of ε and are periodically distributed according to the period εY. Problem (1.8) is then written as follows:

$$\begin{cases} \text{Find } u^\varepsilon \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma\Big(\frac{x}{\varepsilon}\Big) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx, \quad \forall v \in H. \end{cases} \tag{1.10}$$

Figure 2 shows the periodic structure of Ω together with the basic period Y. In this simple model, exactly two scales characterize the structure, the macroscopic scale x and the microscopic one $\frac{x}{\varepsilon}$, describing the micro-oscillations.

The discontinuities of this problem make the model somewhat difficult to treat within the classical theory of partial differential equations as well as from the numerical point of view. Also, the pointwise knowledge of the characteristic of the material does not provide any information on its the global behavior in a simple way.

In this setting, many natural questions arise:

(i) does the temperature u^ε converge to some limit function u^0?
(ii) if so, does u^0 solve some limit boundary value problem?
(ii) are then the coefficients of the limit problem constant?
(iv) finally, is u^0 a good approximation of u^ε?

These questions are very important in the applications since, if one can give positive answers, then the limit coefficients, as it is expected by engineers and physicists, are good approximations of the global characteristics of the composite material, when regarded as an homogeneous one. Moreover, replacing the problem by the limit one, allows to make easy numerical computations.

Let us give the main steps of the procedure, and indicate the main difficulties.

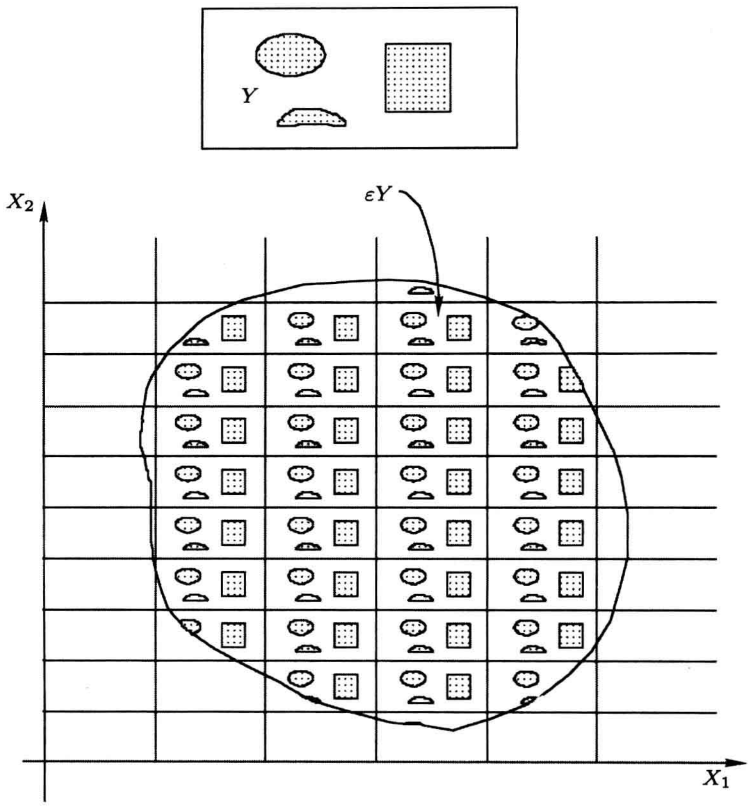

Figure 2

The first point is that the function γ^ε converges in a weak sense to the mean value of γ , i.e. one has

$$\int_\Omega \gamma^\varepsilon(x)v(x)dx \longrightarrow \int_\Omega \mathcal{M}_Y(\gamma)v(x)dx, \tag{1.11}$$

for any integrable function v (cf. Theorem 1.1 below). Here the mean value $\mathcal{M}_Y(f)$ is defined by

$$\mathcal{M}_Y(\gamma) = \frac{1}{|Y|}\int_Y \gamma(y)dy.$$

Next, one can also use weak compactness in the space H to show that, up to a subsequence u^ε converges to some function u^0 and that ∇u^ε weakly converges to ∇u^0. The use of subsequences (which are given by weak compctness arguments) is will not really be a cause for trouble, because (at least in many cases including here) the limit problem obtained has a unique solution. Therefore, at the end of the proof, the whole sequence will converge.

The question is whether this is sufficient to homogenize problem (1.10). To do so, one has to pass to the limit in the product $\xi^\varepsilon \doteq \gamma^\varepsilon \nabla u^\varepsilon$. **This is the main difficulty in the homogenization theory**. In general, the product of two weakly convergent sequences does not converge to the product of the weak limit. But, (up to a subsequence), there is a vector function ξ^0, which is the weak limit of the product $\gamma^\varepsilon \nabla u^\varepsilon$. It is easily proved then that this function ξ^0 satisfies the equation

$$-\text{div}\xi^0 = f. \tag{1.12}$$

But since

$$\xi^0 \neq \mathcal{M}_Y(\gamma)\nabla u^0,$$

from (1.12) one cannot deduce an equation satisfied by u^0.

This already occurs in the one-dimensional case where Ω is some interval (d_1, d_2). Here, one can carry out the full computation explicitely. Since, in one dimension, the divergence is the derivative, it follows from (1.8) or (1.10) that ξ^ε is bounded in $H^1(d_1, d_2)$ hence strongly compact in $L^2(d_1, d_2)$. Up to a subsequence, one can assume that ξ^ε converges strongly to some ξ^0 in the latter space, so that

$$\frac{du^\varepsilon}{dx} = \frac{1}{\gamma^\varepsilon}\xi^\varepsilon \rightharpoonup \mathcal{M}_Y\left(\frac{1}{\gamma}\right)\xi^0 = \frac{du^0}{dx}, \quad \text{thus } \xi^0 = \frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)}\frac{du^0}{dx}.$$

It then follows that u^0 is the unique solution of the homogenized problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)}\dfrac{du^0}{dx}\right) = f \quad \text{in }]d_1, d_2[, \\ u^0(d_1) \doteq u^0(d_2) = 0. \end{cases}$$

Clearly, $\xi^0 \neq \mathcal{M}_Y(\gamma)\nabla u^0$, since

$$\frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \neq \mathcal{M}_Y(\gamma),$$

unless γ is constant (but then, there is no composite!).

Therefore, even for the one-dimensional case this homogenization result is not completely trivial. The situation is of course, more complicated in the general N-dimensional case. The one-dimensional result could suggest that in the N-dimensional case the limit problem can be described in terms of the mean value of γ^{-1}. This is not true, as it can already be seen in the case of layered materials where γ depends only on one variable, say x_1. Then, the homogenized problem of (1.10) is

$$\begin{cases} -\operatorname{div}(A^0 \nabla u^0) = f \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where the homogenized matrix A^0 is constant, diagonal and given by

$$A^0 = \begin{bmatrix} \dfrac{1}{\mathcal{M}_Y(\gamma^{-1})} & 0 & \dots & 0 \\ 0 & \mathcal{M}_Y(\gamma) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_Y(\gamma) \end{bmatrix}.$$

This can be seen as the rule of composition of resistances (inverses of the conductances γ^ε) in series in the first direction, and in parallel in the other directions.

Note that the homogenized material is no more isotropic, since A^0 is not of the form $a^0 I$.

We will see that for the general N-dimensional case, the homogenized problem is still of the form (1.13). The coefficients of A^0 are defined by means of some periodic functions which are the solutions of boundary value problems of the same type as (1.10), but posed in the reference cell Y and with periodic boundary conditions. More precisely,

$$a^0_{ij} = \frac{1}{|Y|}\int_Y \gamma\, \delta_{ij} dy - \frac{1}{|Y|}\int_Y \gamma \frac{\partial \chi_j}{\partial y_i} dy, \quad \forall\, i,j = 1,\dots,N, \tag{1.14}$$

where δ_{ij} is the Kronecker symbol. For each $j = 1,\dots,N$, the function χ_j is the solution of the so-called "cell-problem"

$$\begin{cases} -\operatorname{div}(\gamma(y)\nabla\chi_j) = -\dfrac{\partial\gamma}{\partial y_j} \quad \text{in } Y, \\ \chi_j \quad Y\text{-periodic}, \\ \mathcal{M}_Y(\chi_j) = 0. \end{cases} \tag{1.15}$$

This result can be proved by various methods, which will be sketched in the following section.

In all of them, ε-periodic functions play an important role, and their strong or weak convergences are fundamental. In this respect, let us state the most general result concerning such functions. It will be stated in the whole of $\mathbb{R}^N$ and can be localized in any/every bounded domain Ω.

Theorem 1.1. *Let $\{F_\varepsilon\}_\varepsilon$ be a sequence of measurable functions on the period Y, and set $f_\varepsilon(x) = F_\varepsilon(\frac{x}{\varepsilon})$, defined on $\mathbb{R}^N$. Then*

(i) *for any $p \in [1, \infty]$, the sequence $\{f_\varepsilon\}_\varepsilon$ is bounded in $L^p_{loc}(\mathbb{R}^N)$ (hence in any/every $L^p(\Omega)$ for Ω bounded in $\mathbb{R}^N$) if and only if the sequence $\{F_\varepsilon\}_\varepsilon$ is bounded in $L^p(Y)$;*

(ii) *for any $p \in [1, \infty]$, the sequence $\{f_\varepsilon\}_\varepsilon$ is weakly convergent in $L^p_{loc}(\mathbb{R}^N)$ (hence in any/every $L^p(\Omega)$ for Ω bounded in $\mathbb{R}^N$) if and only if the sequence $\{\mathcal{M}_Y(F_\varepsilon)\}_\varepsilon$ converges in $\mathbb{R}$ (for $p = \infty$, weak convergence is to be replaced by weak-$*$ convergence); in such a case, the weak (or weak$*$) limit of $\{f_\varepsilon\}_\varepsilon$ is the constant function $\lim_{\varepsilon \to 0} \mathcal{M}_Y(\{F_\varepsilon\})$;*

(iii) *for any $p \in [1, \infty[$, the sequence $\{f_\varepsilon\}_\varepsilon$ is strongly convergent in $L^p_{loc}(\mathbb{R}^N)$ (hence in any/every $L^p(\Omega)$ for Ω bounded in $\mathbb{R}^N$) if and only if the sequence $\{F_\varepsilon\}_\varepsilon$ converges strongly to a constant in $L^p(Y)$; in such a case, the strong limit of $\{f_\varepsilon\}_\varepsilon$ is that very constant.*

The proof of Theorem 1.1 is a mere corollary in the Periodic Unfolding method, which is presented in another course.

The presentation of this introductory course is largely inspired by the book of D. Cioranescu and P. Donato, [5]. It is recommended reading. We also refer to its Bibliography for an extensive list of references on the subject up to its publication date.

Another book which is older but still of great interest is that of A. Bensoussan, J.-L. Lions and G. Papanicolaou [2]. For two generalizations which we will not cover, we refer to two books and their bibliography. The first one is [3], an introduction for the method of Γ-convergence, where it is applied to the case of non-quadratic energies. The second is [6], which considers the specific case of reticulated structures.

2 The main ideas of Homogenization

As seen in the above brief description, the theory of homogenization is a way to approach the study of problems with rapidly oscillating coefficients, approximating them by problems with smoother coefficients. It basically goes in three steps.

The three steps of Homogenization

1) To embed the original problem in a family of similar problems indexed by a small parameter "ε". The physical problem corresponds to some specific value, say ε_0 of ε. Corresponding to each $\varepsilon > 0$, there may be one or many solutions u^ε.