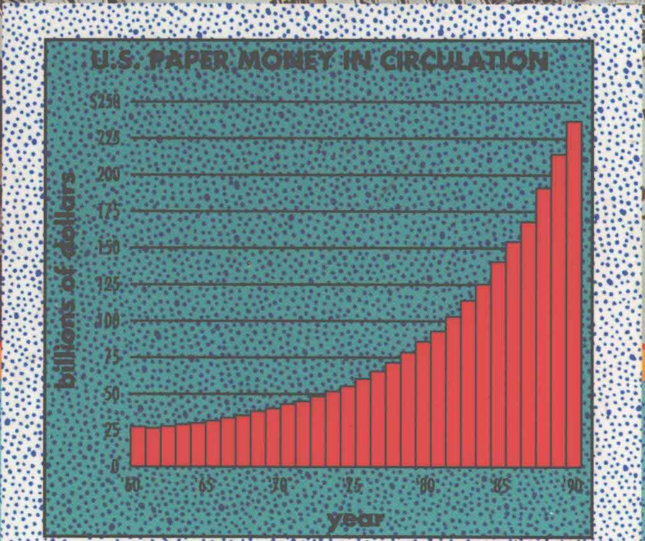
# *Functional* CALCULUS

*Brief Calculus for Management, Life, and Social Sciences*

William C. Ramaley

# Functional Calculus

## Brief Calculus for Management, Life, and Social Sciences

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# PREFACE

## To the Student and Instructor

Isaac Newton created calculus, in part, because he was fascinated by its applications. Today's students are also interested in finding such applications.

**Goals of the Book.** This book provides a brief calculus course that focuses on applications. All decisions about what material to include in this book were guided by asking: Suppose a student is majoring in business, economics, management science, or a biological or social science. What aspects of calculus should this student know? What are the applications of calculus to this student's area of interest? This book attempts to arouse students' *mathematical interests* while developing *mathematical abilities.*

Every effort has been made to present realistic, practical, and interesting problems as they might arise in the world of business meetings, boardrooms, and laboratories. Enabling students to use calculus to discuss and resolve problems is the essential goal of this book.

**Goals of Students.** Students want to be able to find, within themselves, the confidence to proceed in a situation in which mathematical formulations and approaches may be helpful. They want to know some worthwhile and useful ideas about how to begin a problem and where they can find additional help. This book is written to help them achieve their goals.

**Examples.** Each example and application has been selected carefully to provide just enough detail to make the problem clear and meaningful, and to avoid making it overwhelmingly complex. Examples are offered in a lively form, which students can understand, handle, and become successful at doing. It is important for students to realize that many of life's complications can be made simpler and more manageable by a gradual analytic approach.

As the topics of each section are gradually introduced, they are illustrated by two types of examples. First, there is the traditional "fully worked out" example. Following that example is often an additional example, called a "Practice Exercise," whose detailed solution is postponed to the end of the section. This provides students the opportunity to work on the practice exercise before being shown all the steps. Involvement improves learning and retention.

**Informal Mathematics.** Mathematical formulas and results are presented in an informal manner, but never in a misleading or incorrect form. In some situations, a more formal discussion may occur in an (optional) appendix to the section.

**Complex Problems and Calculators.** Problems arising from actual situations rarely have "nice" answers, with small and simple numbers. Naturally, any textbook must contain many "nice" problems when a new concept is being introduced so that the concept itself is not obscured. However, if all the problems were of this sort, students might get an incorrect view of the applicability of mathematics. If we consider a real-world application, the chance that it is "nice" is probably less than the chance of our finding a twenty-dollar bill while walking to lunch. Of course, we should keep our eyes open, but we should be prepared for some problems that have "messy" answers. To do these problems, we need to have a calculator and to learn how to use it. A graphing calculator is optional, but there are many examples and some problems that use them.

**Section Problems.** Problem sets begin with "Foundations" problems, which illustrate the basic skills required to do the exercises. These, in turn, begin with questions to develop students' skills and confidence. Following these problems are more interesting applications and situations. At the end of most sections are problems that ask students to write and discuss some mathematical ideas, and there are problems using graphing calculators. Furthermore, there are "Enrichment Problems," intended for students more interested in the topic and willing to explore the material at a richer level.

**Functions and Other Introductory Topics.** Chapter 1 covers topics some students may have encountered before, but this chapter introduces some terminology used in business and other applications. The chapter may be omitted because the essential terms are reintroduced later. However, the presence of these definitions and their use in applications keeps Chapter 1 from being merely a boring review of previous work. Furthermore, most students need a gradual reintroduction to the variety of functions in that chapter. An understanding of functions forms the bedrock of calculus.

**Word Problems.** A special comment on word problems is warranted: Most instructors and students approach word or story problems with an attitude that might be expressed as, "Don't pay any attention to the words at all. Get to the essentials. Look for key words and phrases. The words are just window dressing." On one hand, that attitude is exactly right, and we do need to develop the skills to dive right into the problem, see through the setting, and firmly grasp the essentials. On the other hand, students need to realize that the mathematics they are learning does have real-world applications. Under all the complex, full, and beautiful things we see is a hidden mathematical framework, a scaffold to support the applications. Many instructors of mathematics take for granted the existence and intrinsic interest of this mathematical scaffold, but students may not.

**Historical Comments.** This book contains many historical comments. In these comments, a special effort has been made to point out those contributions made by mathematicians to solve applied problems of interest to society. We may not find it easier to find a derivative just because we learn the two creators of calculus, Newton and Leibniz, both directed their countries' mints for coining money, but we may see mathematicians as being more human. Furthermore, such knowledge allows us to feel less removed from mathematicians and doing mathematics ourselves. Lastly, historical comments are often just plain fun.

## Especially to the Instructor

This book may be used in several ways. For a three or four credit-hour semester course, you may use the first six chapters (omitting the starred sections), plus sections 7.1–7.3. If that seems to be too much material, I would urge at least a mention of the idea of optimization of a function of two variables. A longer or more intensive course could cover all of the first six chapters, plus most of Chapter 7.

For a class well versed in functions, Chapter 1 may be covered swiftly as a review. However, the many applications there are of interest even to students who have been exposed to the mathematical material previously.

## Especially to the Student

**The Book.** This text is meant to be *used by you.* It is more than merely a collection of topics and problems. It is an essential resource. The effort that you and your instructor put into this course will be connected together by this book, for it contains explanations of ideas as well as worked examples and problems. All of these will help you understand calculus and the many uses of calculus concepts.

**Answers.** Students are often intimidated by the fact that most problems at this level of mathematics have just one answer. However—and this "however" is very important—very often there are different approaches to arrive at that answer. Thus, your way of solving a problem may seem (or even actually be) different from the way shown in the book or by your instructor. This does not mean that one particular way is always the only correct way. In fact, if you can work a problem by several methods, then you have a check on your work.

The answers given in the back of the book are often presented in both exact and approximate forms. The reason for this is that the world outside the classroom uses decimal approximations. We should also note that there are often several equivalent forms for an answer. For example, you might obtain an answer of $\sqrt{20}$ and find $2\sqrt{5}$ as the answer in the back. If your answer differs from the answer given, do not abandon your answer immediately. Take a moment to see if the answers are equivalent.

**Evaluating Answers.** When at last you arrive at an answer, ask yourself, "Does this answer make sense?" Try to understand what you have done. In the real world, you do not want to be telling the filling station operator that you have just pumped 137 gallons of gas into a car that has a 20-gallon tank, when you meant 13.7 gallons. It is very important to try to get some "feel" for reasonable answers. That can only be done by working many, many problems.

**Success.** I expect that as you work through this book, there will be times when you ask, "What does he mean by this?" or "How did he get that?" I hope those times are few and I wish I could be there to try to explain what I did mean, or to work through another example, or to try a different explanation. But please do have some patience with yourself. Thousands of students have learned this material successfully and *you can too.*

## Acknowledgments

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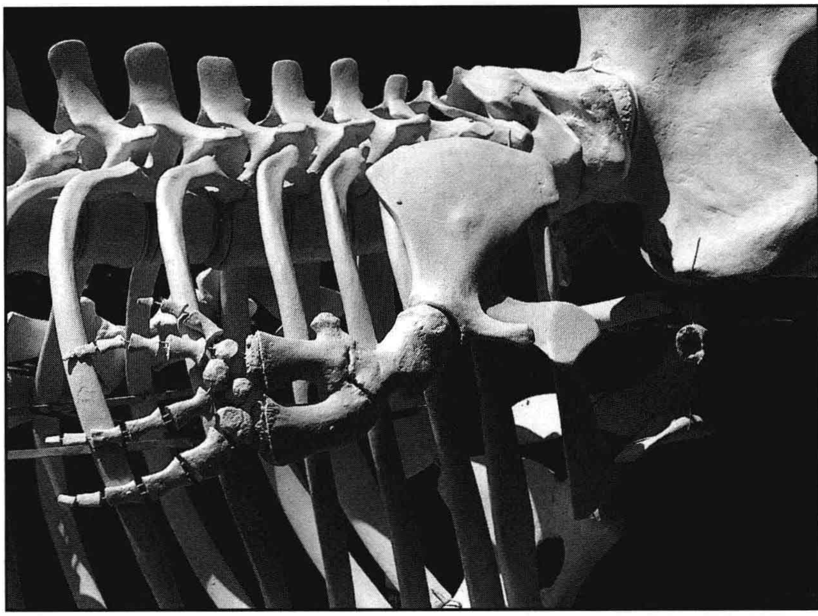
The * denotes an optional section.



CHAPTER 1

# NUMBERS AND FUNCTIONS



THE TOPICS OF THIS CHAPTER PROVIDE THE NECESSARY FOUNDATION FOR building calculus. Much of the material may be familiar, but its review will serve two purposes. First, its study provides a common background and momentum for studying calculus. Second, we will see and use these mathematical ideas to create models of the real world. Our mathematical models will apply to income taxes, whale skeletons, cardiac output, the Golden Gate Bridge, and beyond.

## 1.1 REAL NUMBERS, SETS, AND INEQUALITIES

In this section we discuss real numbers, inequalities, and absolute values, and then present specifications for sets in several ways. In particular, we introduce sets that are intervals of numbers. One use of intervals is to describe the solutions of inequalities.

## Mathematical Innovators
### *Fibonacci of Pisa (1175–1250)*

In our everyday computations we are comfortable using both positive and negative numbers. However, as recently as the thirteenth century only positive numbers were allowed as solutions to problems. Writers and thinkers would have asked, "What could be less than nothing?"

The use of negative numbers first appeared in the book *Flos,* written about 1225 by Fibonacci. He was explaining a business problem and interpreted a negative number as representing a financial loss.

Fibonacci, the son of a merchant, was born about 1175 in Pisa, Italy. He studied while traveling throughout Northern Africa and countries bordering the eastern Mediterranean Sea. There he learned the positional system of notation, which we use today, with its concept of "zero." His fellow merchants found it difficult to accept the idea that "zero" was something that could stand for nothing. However, gradually they saw the advantages of writing a number such as 1202 in that form, rather than in the Roman numeral form of MCCII.

## Real Numbers

The only numbers we are going to discuss in this book are real numbers, so everywhere in this book the word *number* means "real number." Sometimes these numbers are referred to as "the reals."

The term *system of real numbers* refers not only to the numbers themselves, but also to the arithmetical properties of addition, subtraction, multiplication, and division, all of which allow us to operate on numbers to create new numbers.

The system of real numbers is founded on the *integers:*

$$\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots$$

Real numbers that can be expressed as ratios of integers, such as $-5/2$ or $3/4$, are called *rational numbers.* A rational number expressed in decimal form either terminates or produces a repeating block of digits. Furthermore, any terminating decimal or repeating decimal can be expressed as the quotient of two integers. For example, both the terminating decimal 0.75 and the repeating decimal $0.121212 \cdots$, are rational numbers. The first equals $3/4$ and the second equals $4/33$.

In addition, there are real numbers that are not rational numbers. For example, the number represented by $\sqrt{2}$ is an *irrational number.* Calculators may lead us to believe that all numbers are decimal numbers that terminate. For example, by entering $\boxed{2}$ and then pressing the $\boxed{\sqrt{\ }}$ key on a calculator, we may be tempted to write $\sqrt{2} = 1.414213562$. However, $\sqrt{2}$ is not exactly equal to 1.414213562. The calculator is simply rounding off the value of $\sqrt{2}$ to nine decimal places.

There are situations in which we need an exact value and other situations for which approximations are satisfactory. In fact, many times the approximate solution is more useful. For example, when asked whether $\sqrt{2}$ or 13/9 is larger, most people decide by using decimal approximations for each. Furthermore, the world beyond the classroom almost always uses decimal approximations.

Numbers that are quite large or quite small are expressed in *scientific notation.* On a calculator, part of the display of the number will be the power of 10 by which the displayed value should be multiplied. For instance, check that on your calculator the number 1/1996 is displayed as $\boxed{5.01002^{-04}}$ or $\boxed{5.01002\text{ E }-04}$ or something equivalent. This display means the number is approximately $5.01002 \cdot 10^{-4} = 0.000501002$.

In this book, we usually use four places to the right of the decimal point as our approximation. We say that $\pi$ is about 3.1416, written as $\pi \approx 3.1416$, or that $\sqrt{2}$ is about 1.4142. If the number is expressed in scientific notation, then we use four places to the right of the displayed decimal point. This rule is only a guideline; some situations require more precision than others.

Using all of the digits shown by our calculator may give quite an incorrect impression. For example, suppose someone is told that a swimming pool is 25 meters long and they want to know how many feet that is. A standard formula used for the conversion of meters to feet is to multiply the number of meters by 3.281. Doing so would give a length of $25 \cdot 3.281 = 82.025$ feet. However, it is quite misleading to say the pool is 82.025 feet long, because to do so implies that we know the length to the nearest thousandth of a foot. Quite likely, the length in meters is only known to the nearest tenth, or maybe hundredth, of a meter.

When we are performing calculations, we should not use any approximation until we need to do so. For example, suppose we were to use 4.4 as an approximation for $\sqrt{19}$ and 1.7 as an approximation for $\sqrt{3}$. Because our calculator displays 4.4/1.7 as being 2.588235294, we might be tempted to use 2.6 as a one-place approximation for $\sqrt{19}/\sqrt{3}$. Actually, to find the correct one-place approximation for $\sqrt{19}/\sqrt{3}$ we should find 4.358898944 divided by 1.7320508, which the calculator displays as 2.516611478, and which we would round off to 2.5.

Often we wish to transform a number or answer into an equivalent form. For example, the process known as *rationalization* changes $1/\sqrt{2}$ into $\sqrt{2}/2$ by multiplying the numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The knowledge and skills needed to transform different forms of mathematical expressions are precisely some of the most valuable aspects of mathematics, and we devote a considerable amount of effort to acquiring this knowledge. However, sometimes such transformations become distracting. We should always consider the time and effort involved in finding equivalent exact answers and whether such an effort is appropriate.

To illustrate the situation, consider the expressions $1/\sqrt{2}$ and $\sqrt{2}/2$. The form $\sqrt{2}/2$ was preferred at a time when long divisions were performed by hand because it was far easier to divide 1.4142 by 2 than to divide 1 by 1.4142. However, using calculators to obtain an approximation, we use *fewer* keystrokes to evaluate $1/\sqrt{2}$ than to evaluate $\sqrt{2}/2$. As a consequence, if a decimal approximation is desired, we should not transform $1/\sqrt{2}$ into $\sqrt{2}/2$ before finding the approximation.

## The Real Number Line

A convenient representation of all of the real numbers is to associate each number with exactly one point on a horizontal straight line. To do this, we choose an arbitrary point and assign it to the number 0. That point is called the *origin* or the *zero-point.* Choose another point, to the right of the zero-point, and assign it to the number 1. Using the distance between the 0 and the 1 as our measure of one unit of distance, we assign all the points to the right of 0 to the positive numbers and all the points to the left to the negative numbers. A line so marked is a *real number line* (figure 1.1.1), and the numbers are the *coordinates* of the points.

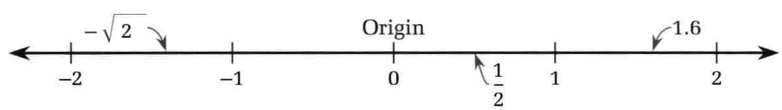



**FIGURE 1.1.1**

## Inequalities

DEFINITION

### Greater than and less than

If the point on the line that corresponds to the number $a$ is to the left of the point that corresponds to the number $b$, then we say *$a$ is less than $b$* and write: $a < b$. If the point for $a$ is to the right of the point for $b$, we say that *$a$ is greater than $b$* and we write: $a > b$. We write $a \le b$ when *$a$ is less than or equal to $b$.* Similarly, $a \ge b$ means *$a$ is greater than or equal to $b$.*

Expressions such as $3 \le 5$, $6 > 2$, and $-3 \le -1$ are examples of *inequalities*. At our convenience, we write inequalities in either direction. For example, to express the relationship between 3 and 5 we may think of 3 as being less than 5 and so write $3 < 5$. On the other hand, it is equivalently true that 5 is greater than 3, and so we can write $5 > 3$.

Shown in figure 1.1.2 are $5 < 7$, $3 \le 7$, $3 \ge -1$, and $4 \le 4$.

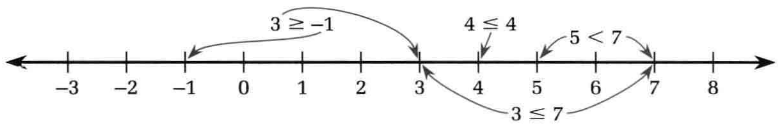



**FIGURE 1.1.2**

If we write a double inequality, such as $a < b \le c$, we mean that *both* $a < b$ and $b \le c$ have to be true. For example, $3 < x \le 5$ means that $x$ must be greater than 3 but no greater than 5.

**WARNING**

When writing double inequalities be careful with notation. For example, consider the double inequality $1 < x < -2$. This says that $x$ is a number such that *both* $1 < x$ and $x < -2$ have to be true. There are no such values for $x$. If $x$ was meant to be a real number that could be greater than 1 or could be less than $-2$, then one correct way of writing this would be $1 < x$ or $x < -2$.

The following properties of inequalities allow us to perform arithmetic with inequalities. Similar rules hold if "<" is replaced by "≤" and ">" is replaced by "≥."

## INEQUALITY PROPERTIES

For any real numbers $a$, $b$, and $c$:

**(i)** If $a < b$, then $a + c < b + c$ and $a - c < b - c$

**(ii)** If $a < b$ and $c > 0$, then $ac < bc$ and $a/c < b/c$

**(iii)** If $a < b$ and $c < 0$, then $ac > bc$ and $a/c > b/c$

Notice that properties (ii) and (iii) state that the direction of an inequality upon multiplication or division is kept the same when multiplying or dividing by a positive number and the direction is reversed when multiplying or dividing by a negative number.

## EXAMPLE 1 Inequality properties

As illustrations of inequality properties, consider the following examples. In each case we check the correctness of the resulting inequality by actually doing the arithmetic and comparing the resulting numbers. The particular numbers used are chosen as random representatives. Each example illustrates the inequality property with the corresponding number.

Property (i). From knowing $2 < 5$, we have $2 + 3 < 5 + 3$. As a check, we calculate $2 + 3 = 5$ and $5 + 3 = 8$. Indeed, $5 < 8$. From knowing $3 < 7$, we have $3 - 2 < 7 - 2$. Again, we can check by finding $3 - 2 = 1$ and $7 - 2 = 5$ and seeing $1 < 5$.

Property (ii). From $1 < 3$ and $2 > 0$, by multiplying we have $1 \cdot 2 < 3 \cdot 2$ (checking, $2 < 6$) and by dividing we have $1/2 < 3/2$ (checking, $0.5 < 1.5$).

Property (iii). Because $2 < 4$ and $-3 < 0$, we have $2(-3) > 4(-3)$ (checking, $-6 > -12$).

We may think geometrically about any of these examples. For instance, in figure 1.1.3 we can imagine 2 being to the left of 4. Upon multiplication by $-3$ we have $2(-3) = -6$ and $4(-3) = -12$. The point for $-6$ is to the right of $-12$, so $-6 > -12$.

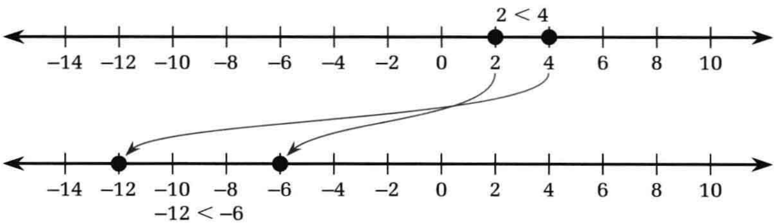



**FIGURE 1.1.3**

## Absolute Value

Sometimes we only want, or need, to know the size of a number. This size is determined by the distance between its associated point and the zero point and it is the *absolute value* of the number.

DEFINITION

## The absolute value

The *absolute value* of a number $a$ is written as $|a|$ and is equal to $a$ if $a$ is positive or zero and is equal to $-a$ when $a$ is negative. In symbols,

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

The second part of the definition, which says that $|a|$ is equal to $-a$ when $a$ is negative, may seem difficult to understand because we always want the absolute value to be positive. But keep in mind that if we are finding the absolute value of a negative number such as $-3.5$, the $a$ in the definition is the entire negative number, $-3.5$, not just the 3.5.

Hence, with $a = -3.5$, then $|a| = -a$ gives us $-(a) = -(-3.5) = 3.5$.

## EXAMPLE 2 Absolute values

Evaluate $|3|$, $|5.4|$, $|-3|$, $|-0.7|$, and $|x - 2|$.

### SOLUTION

$|3| = 3$, $|5.4| = 5.4$, $|-3| = -(-3) = 3$, and $|-0.7| = 0.7$. To evaluate the expression $|x - 2|$ we need to know more about $x$. If $x - 2 \geq 0$, which is equivalent to $x \geq 2$, then $|x - 2| = x - 2$. However, if $x - 2 < 0$, which is equivalent to $x < 2$, then $|x - 2| = -(x - 2) = -x + 2$. Graphically, we can represent this in figure 1.1.4.

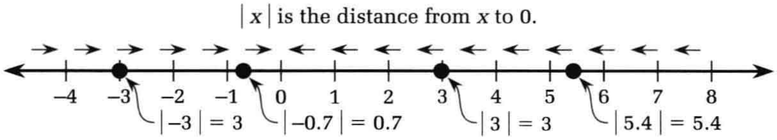



**FIGURE 1.1.4**

## PRACTICE EXERCISE 1

Find the absolute values of the following: 5, $\sqrt{3}$, $-2$, $-0.4$, $x + 3$.

The following example is an application of the concept of absolute value that is of interest to economists and sociologists. It uses absolute values to determine a comparative distribution of two resources.

## EXAMPLE 3 Application to economics

### Coefficient of Concentration

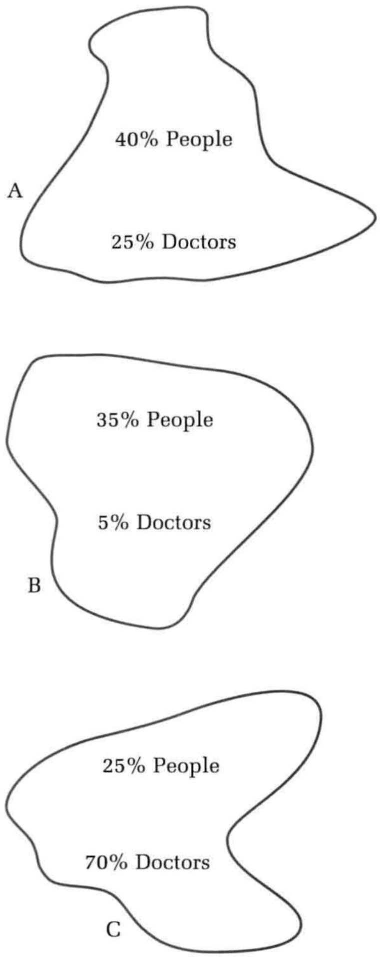


**FIGURE 1.1.5**

Economists define the *coefficient of concentration, r,* to be equal to the sum of all the absolute differences between percentages, written decimally, of two resources.

For example, suppose there are three regions, A, B, and C, as shown in figure 1.1.5. Region A contains 25% of all the doctors and 40% of all the people; region B contains 5% of all the doctors and 35% of all the people; and region C contains 70% of all the doctors and 25% of all the people. The coefficient of concentration is:

$$\begin{aligned} r &= |0.25 - 0.40| + |0.05 - 0.35| + |0.70 - 0.25| \\ &= |-0.15| + |-0.30| + |0.45| = 0.9 \end{aligned}$$

A coefficient of concentration that is near zero means that the resources under comparison are distributed comparably equally. That is, continuing with the situation in Example 3, if $r$ was near zero, then the doctors would be located where the people were.

The following properties of absolute values can be proved.

**PROPERTIES OF ABSOLUTE VALUES**

For all real numbers $a$ and $b$

**(i)** $|a| \geq 0$

**(ii)** $|a \cdot b| = |a| \cdot |b|$

**(iii)** $|a| = |-a|$

**(iv)** $|a + b| \leq |a| + |b|$

It is often convenient to change an inequality that involves an absolute value into an inequality that does not. For example, if we wish to find values of $x$ for which $|x| < 6$, we may prefer to rewrite the condition to read: find values of $x$ for which $-6 < x < 6$. These are equivalent conditions on the possible values for $x$.

The following are equivalent conditions on the possible values for the number $x$. (Remember, we are using the phrase "any number" to mean "any real number.")