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HANDBOOK OF MATHEMATICAL LOGIC

EDITED BY
JON BARWISE

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WITH THE COOPERATION OF
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Foreword

The *Handbook of Mathematical Logic* is an attempt to share with the entire mathematical community some modern developments in logic. We have selected from the wealth of topics available *some* of those which deal with the basic concerns of the subject, or are particularly important for applications to other parts of mathematics, or both.

Mathematical logic is traditionally divided into four parts: model theory, set theory, recursion theory and proof theory. We have followed this division, for lack of a better one, in arranging this book. It made the placement of chapters where there is interaction of several parts of logic a difficult matter, so the division should be taken with a grain of salt. Each of the four parts begins with a short guide to the chapters that follow. The first chapter or two in each part are introductory in scope. More advanced chapters follow, as do chapters on applied or applicable parts of mathematical logic. Each chapter is definitely written for someone who is not a specialist in the field in question. On the other hand, each chapter has its own intended audience which varies from chapter to chapter. In particular, there are some chapters which are not written for the general mathematician, but rather are aimed at logicians in one field by logicians in another.

We hope that many mathematicians will pick up this book out of idle curiosity and leaf through it to get a feeling for what is going on in another part of mathematics. It is hard to imagine a mathematician who could spend ten minutes doing this without wanting to pursue a few chapters, and the introductory sections of others, in some detail. It is an opportunity that hasn't existed before and is the reason for the Handbook.

JON BARWISE

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Table of Contents

Foreword vii

Contributors viii

PART A: MODEL THEORY

Guide to Part A 3

A.1. An introduction to first-order logic, *Jon Barwise* 5

A.2. Fundamentals of model theory, *H. Jerome Keisler* 47

A.3. Ultraproducts for algebraists, *Paul C. Eklof* 105

A.4. Model completeness, *Angus Macintyre* 139

A.5. Homogenous sets, *Michael Morely* 181

A.6. Infinitesimal analysis of curves and surfaces, *K. D. Stroyan* . . 197

A.7. Admissible sets and infinitary logic, *M. Makkai* 233

A.8. Doctrines in categorical logic, *A. Kock* and *G. E. Reyes* . . . 283

PART B: SET THEORY

Guide to Part B 317

B.1. Axioms of set theory, *J.R. Shoenfield* 321

B.2. About the axiom of choice, *Thomas J. Jech* 345

B.3. Combinatorics, *Kenneth Kunen* 371

B.4. Forcing, *John P. Burgess* 403

B.5. Constructibility, *Keith J. Devlin* 453

B.6. Martin's Axiom, *Mary Ellen Rudin* 491

B.7. Consistency results in topology, *I. Juhász* 503

PART C: RECURSION THEORY

Guide to Part C 525

C.1. Elements of recursion theory, *Herbert B. Enderton* 527

C.2. Unsolvable problems, *Martin Davis* 567

C.3. Decidable theories, *Michael O. Rabin* 595

C.4. Degrees of unsolvability: a survey of results, *Stephen G. Simpson* 631

C.5. α-recursion theory, *Richard A. Shore* 653

C.6. Recursion in higher types, *Alexander Kechris and Yiannis N. Moschovakis* 681

C.7. An introduction to inductive definitions, *Peter Aczel* 739

C.8. Descriptive set theory: Projective sets, *Donald A. Martin* . . . 783

PART D: PROOF THEORY AND CONSTRUCTIVE MATHEMATICS

Guide to Part D 819

D.1. The incompleteness theorems, *C. Smorynski* 821

D.2. Proof theory: Some applications of cut-elimination, *Helmut Schwichtenberg* 867

D.3. Herbrand's Theorem and Gentzen's notion of a direct proof, *Richard Statman* 897

D.4. Theories of finite type related to mathematical practice, *Solomon Feferman* 913

D.5. Aspects of constructive mathematics, *A. S. Troelstra* 973

D.6. The logic of topoi, *Michael P. Fourman* 1053

D.7. The type free lambda calculus, *Henk Barendregt* 1091

D.8. A mathematical incompleteness in Peano Arithmetic, *Jeff Paris and Leo Harrington* 1133

Author Index 1143

Subject Index 1151

PART A

Model Theory

Guide to Part A: *Model Theory*

with the cooperation of
H. J. KEISLER

A.1. BARWISE
An introduction to first-order logic 5
A.2. KEISLER
Fundamentals of model theory 47
A.3. EKLOF
Ultraproducts for algebraists 105
A.4. MACINTYRE
Model completeness 139
A.5. MORLEY
Homogenous sets 181
A.6. STROYAN
Infinitesimal analysis of curves and surfaces 197
A.7. MAKKAI
Admissible sets and infinitary logic 233
A.8. KOCK and REYES
Doctrines in categorical logic 283

This part of the Handbook is concerned with the fundamental relationship between mathematical statements (axioms), on the one hand, and mathematical structures (models) which satisfy them, on the other. The emphasis is on the model theory of *first-order* statements. Barwise's chapter, written for those with no prior knowledge of first-order logic, explains the most basic notions. This material is really pre-model theory, and is needed for most of the chapters in the Handbook.

Keisler's chapter contains the real introduction to model theory. By glancing through this chapter the reader can see the concerns of the subject illustrated with basic results and applications. The next three chapters treat important topics from model theory in depth and are aimed more at the algebraist.

Eklof's chapter discusses the ultraproduct operation, its relation with first-order logic, and its positive applications to algebra. Macintyre's chapter discusses both positive and negative applications to algebra of Abraham Robinson's notion of *model complete theory* and related concepts of "algebraically closed".

Morley's chapter on homogenous sets discusses so-called Ehrenfeucht–Mostowski models. This construction has proven extremely useful in model theory and in applications to set theory. It has had some applications to other parts of mathematics, but should have more once it becomes better known.

To date the principal application of model theory outside algebra and set theory comes from Robinson's "nonstandard analysis". Stroyan's chapter discusses elementary aspects of the subject and gives a more advanced case study of the hidden role infinitesimals play in differential geometry.

The last three chapters in Part A go beyond ordinary first-order logic. Some extensions of first-order logic are mentioned in the last section of Barwise's chapter and discussed in more detail in the last section of Keisler's chapter. Of all the known extensions, the logic $L_{\omega_1\omega}$ has the smoothest model theory. This logic, and its admissible fragments, are discussed in Makkai's chapter.

The final chapter, by Kock and Reyes, is quite different in character. It gives the category theoretical point of view of some topics from model theory and other parts of logic.

It was planned to have a chapter on stability theory and one on abstract model theory. This proved impossible so stability theory is now surveyed in Section 8 of Keisler's chapter. Abstract model theory is discussed at the end of Barwise's chapter and is touched on in Keisler's chapter. Among the other chapters of the Handbook which are particularly relevant to model theory are Rabin's chapter on decidable and undecidable theories, and Aczel's chapter on inductive definitions, both in Part C of the book.

A.1

An Introduction to First-Order Logic

JON BARWISE*

Contents

1. Foreword 6
2. How to tell if you are in the realm of first-order logic 6
3. The formalization of first-order logic 17
4. The Completeness Theorem 22
5. Beyond first-order logic 41
References 45

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HANDBOOK OF MATHEMATICAL LOGIC
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1. Foreword

This introductory chapter is written for the (less than ideal) mathematician who knows next to nothing about mathematical logic, and is entirely expository in nature. This might be someone who tried to read a later chapter but got bogged down simply because he did not understand the basic notions. For most readers a quick reading of Section 2 and the introductions to Sections 4 and 5 should suffice.

Modern mathematics might be described as the science of abstract objects, be they real numbers, functions, surfaces, algebraic structures or whatever. Mathematical logic adds a new dimension to this science by paying attention to the language used in mathematics, to the ways abstract objects are defined, and to the laws of logic which govern us as we reason about these objects. The logician undertakes this study with the hope of understanding the phenomena of mathematical experience and eventually contributing to mathematics, both in terms of important results that arise out of the subject itself (Gödel's Second Incompleteness Theorem is the most famous example) and in terms of applications to other branches of mathematics. The chapters of this Handbook are intended to illustrate both of these aspects of mathematical logic.

Modern mathematical logic has its origins in the dream of Leibniz of a universal symbolic calculus which could encompass all mental activity of a logically rigourous nature, in particular, all of mathematics. This vision was too grandiose for Leibniz to realize. His writings on the subject were largely forgotten and had little influence on the actual course of events. It took Boole, Frege, Peano, Russell and Whitehead, Hilbert, Skolem, Gödel, Tarski and their followers, armed with more powerful abstract methods, and motivated (at least in the case of Russell and Hilbert) by apparent problems in the foundations of mathematics, to realize a significant part of Leibniz' dream.

2. How to tell if you are in the realm of first-order logic

Our goal in this section is quite modest: to give the reader, by means of examples, a feeling for what can and what cannot be expressed in first-order logic. Most of our examples are taken from the wealth of notions in modern algebra with which most mathematicians have at least a nodding acquaintance.

The basic building blocks of first-order logic consist of the logical connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies), the equality symbol $=$, quantifiers $\forall$ (for all), $\exists$ (there exists) plus an infinite sequence of variables $x, y, z, x_1, y_1, \ldots$ and some parentheses), (to help the formulas stay readable.

In addition to these logical symbols, a set L of primitive non-logical symbols is given by the topic under discussion. For example, if we are working with abelian groups then the set L has a function symbol $+$ for group addition and a constant symbol 0 for the zero element. If we are working with orderings, then L has a relation symbol $<$. For the study of set theory, L has a relation symbol $\in$. We will postpone the rather tedious formal definition of formula of first-order logic until the next section. Here we stress only that formulas are certain *finite* strings of symbols.

The "first" in the phrase "first-order logic" is there to distinguish this form of logic from stronger logics (like second-order or weak second-order logic) where certain extralogical notions (like set or natural number) are taken as given in advance. In particular, in first-order logic the quantifiers $\forall$ and $\exists$ always range over elements of the domain M of discourse. By contrast, second-order logic allows one to quantify over subsets of M and functions F mapping, say, $M \times M$ into M. (Third-order logic goes on to sets of functions, etc.) Weak second-order logic allows quantification over finite subsets of M and over natural numbers. There are good reasons for considering first-order logic to be the basic language of mathematics; these will be discussed in Section 5. We assume here that the reader has his own motivation for wanting to find out what first-order logic is.

Group theory

Our first few examples come from group theory. Consider the following notions:

(a) group,
(b) abelian group,
(c) abelian group with every element of order $\leq n$,
(d) divisible group,
(e) torsion-free group,
(f) torsion group.

The notions (a)–(c) are easily axiomatized by a few first-order axioms. Notions (d) and (e) take an infinite list of axioms. The last notion (f) is not first-order. Let's see why.

A group G is a triple $G = \langle G, +, 0\rangle$ (where G is a nonempty set, $0 \in G$ and $+$ is a function mapping $G \times G$ into G) which satisfies the following first-order axioms, or sentences:

$$\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z], \tag{1}$$

$$\forall x\, [x + 0 = x], \tag{2}$$

$$\forall x\, \exists y\, [x + y = 0]. \tag{3}$$

The logician might say that G *is a model of* (1), (2), (3) and write $G \models$ (1), (2), (3), instead of saying that G satisfies (1), (2), (3).

An *abelian group* is a group G satisfying the axiom

$$\forall x\, \forall y\, [x + y = y + x]. \tag{4}$$

The choice of the symbol "+" in (1)–(4) is dictated by convention only; it has no real significance.

To express the next notion we abbreviate the formal term $(x + x)$ by $2x$, the term $((x + x) + x)$ by $3x$ and, by induction, we abbreviate the term $(nx + x)$ by $(n + 1)x$. An abelian group G has every element of order $\leq n$ if G is a model of

$$\forall x\, [x = 0 \vee 2x = 0 \vee \cdots \vee nx = 0]. \tag{5}$$

This is a simple first-order sentence.

An abelian group G is *divisible* if

$$\forall n \geq 1\, \forall x\, \exists y\, [ny = x]. \tag{6}$$

This would count as a sentence of weak second-order logic but it is *not* a first-order axiom because the leading quantifier ranges over the set of positive natural numbers, rather than over the domain of discourse G. We can, however, replace this expression by the following infinite list of axioms:

$$\forall x\, \exists y\, [2y = x], \tag*{$(6)_2$}$$

$$\forall x\, \exists y\, [3y = x], \tag*{$(6)_3$}$$

$$\vdots$$

$$\forall x\, \exists y\, [ny = x], \tag*{$(6)_n$}$$

$$\vdots$$

(We left off $(6)_1$ since it is the trivial sentence $\forall x\, \exists y\, [x = y]$.) For most purposes such an effectively presented infinite list of axioms is practically as good as a finite list. Still, it is worth proving for our own satisfaction that it is not just lack of imagination which forces us to use an infinite list to express the notion.

2.1. PROPOSITION. *Any finite set of first-order sentences true in all divisible abelian groups is true in some nondivisible abelian group.*

In other words, the notion of divisible abelian group is not *finitely* axiomatizable in first-order logic. We delay the proof of this result for a few paragraphs.

We discover essentially the same phenomenon when we attempt to axiomatize the concept of *torsion-free* abelian group:

$$\forall n \geq 1\, \forall x\, [x \neq 0 \rightarrow nx \neq 0]. \tag{7}$$

This sentence of weak second-order logic turns into an infinite list of first-order axioms:

$$\forall x\, [x \neq 0 \rightarrow nx \neq 0]. \qquad (7)_n$$

We have the corresponding negative result.

2.2. Proposition. *The notion of torsion-free abelian group is not finitely axiomatizable in first-order logic.*

An abelian group G is *torsion* if it satisfies

$$\forall x\, \exists n \geq 1\, [nx = 0]. \tag{8}$$

This is a sentence of weak second-order logic but it is not first-order because it has the quantifier $\exists n$ over natural numbers. We could try to imitate (5) but look what happens:

$$\forall x\, [x = 0 \vee 2x = 0 \vee \cdots \vee nx = 0 \vee \cdots]. \qquad (8)'$$

This sort of expression is analogous to an infinite formal power series and the study of such idealized "infinitary formulas" has turned out to be quite profitable (see 5.3, and Chapters A.2 and A.7) but it is not part of ordinary first-order logic. To clinch matters we will prove the following result.

2.3. Proposition. *The set of first-order sentences true in all torsion abelian groups is true in some abelian group H which is not torsion.*

In fact, what we will show is that if G is an abelian group with no finite bound on the order of its elements, then there is a group H which is not torsion but such that $G \equiv H$, which means that every first-order sentence true in G is also true in H, and vice versa. Therefore the class of torsion groups cannot be characterized even by a set of first-order axioms — finite or infinite.