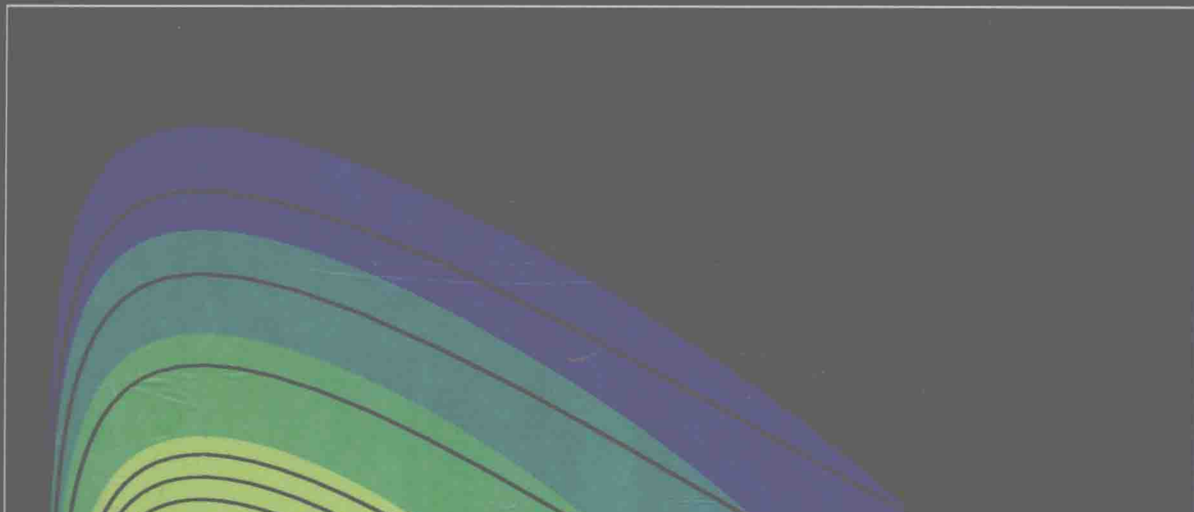
THIRD EDITION

Elementary Differential Equations with Applications



C.H. Edwards, Jr.
David E. Penney

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Elementary Differential Equations

with Applications

To Alice and Carol

Preface

We wrote this book to provide a concrete and readable text for the traditional course in elementary differential equations that science, engineering, and mathematics students take following calculus. It includes enough material appropriately arranged for different courses varying in length from one quarter to two quarters. Our treatment is shaped throughout by the goal of an exposition that students will find accessible, attractive, and interesting. We hope that we have anticipated and addressed most of the questions and difficulties that students typically encounter when they study differential equations for the first time.

The book begins and ends with discussions and examples of the mathematical modeling of real-world situations. The fact that differential equations have diverse and important applications is too familiar for extensive comment here. But these applications have played a singular role in the historical development of this subject. Whole areas of the subject exist mainly because of their applications. So in teaching it, we want our students to learn first to solve those differential equations that enjoy the most frequent and interesting applications.

We therefore make consistent use of appealing applications to motivate and illustrate the standard elementary techniques of solution of differential equations. A number of the more substantial applications are placed in optional sections, each marked with an asterisk (in the table of contents and in the text). These sections can be omitted without loss of continuity, but their availability can provide instructors with flexibility for variations in emphasis.

While according real-world applications their due, we also think the first course in differential equations should be a window on the world of mathematics. Matters of definition, classification, and logical structure deserve (and receive here) careful attention—for the first time in the mathematical experience of many of the students (and perhaps for the last time in some cases). While it is neither feasible nor desirable to include proofs of the fundamental existence and uniqueness theorems along the way in an elementary course, students need to see precise and clear-cut statements of these theorems, and to understand their role in the subject. We include appropriate existence and uniqueness proofs in the Appendix, and occasionally refer to them in the main body of the text.

The list of introductory topics in differential equations is quite standard, and a glance at our chapter titles will reveal no major surprises, although in the fine structure we have attempted to add a bit of zest here and there. A number of different permutations in the order of topics are possible, and the table that follows this preface exhibits the logical dependence between chapters. In most chapters the principal ideas of the topic are introduced in the first few sections of the chapter, and the remaining sections are devoted to extensions and applications. Hence the instructor has a wide range of choice regarding breadth and depth of coverage.

At various points our approach reflects the widespread use of computer programs for the numerical solution of differential equations. Nevertheless, we continue to believe that the traditional elementary analytical methods of solution are important for students to learn. One reason is that effective and reliable use of numerical methods often requires preliminary analysis using standard elementary techniques; the construction of a realistic numerical model often is based on the study of a simpler analytical model.

Third Edition Features

In preparing this revision we have taken advantage of many valuable comments and suggestions from users of the first two editions. In addition to the specific changes mentioned below, we have rewritten many discussions for greater clarity, and have added new remarks, applications, examples, problems, and computational details throughout the book. We hope that the additional computer-generated artwork that we have included will help students to visualize better the geometric aspects of differential equations.

Chapter 1 naturally treats first order equations, with separable equations (Section 1.4), linear equations (Section 1.5), substitution methods (Section 1.6), and exact equations (Section 1.7) comprising the core of the chapter. In order to make the concepts of linear independence and general solutions more concrete and tangible, we discuss only second order equations in Section 2.1, and follow with the nth order case in Section 2.2.

Chapter 3 begins with a review of the basic facts about power series that will be needed. The first three sections of the chapter treat the standard power series techniques for the solution of linear equations with variable coefficients. We devote more attention than usual to certain matters—such as shifting indices of summation—that are mathematically routine but nevertheless troublesome for many students. In Section 3.4 (optional) we include for reference more detail on the method of Frobenius than ordinarily will be covered in the classroom. Similarly, we go slightly further than is customary in Section 3.6 (optional) with applications of Bessel functions. Chapter 4 on Laplace transforms is rather standard, although our discussion in Section 4.6 (optional) of impulses and Dirac delta functions may have some merit.

There is much variation in the treatment of linear systems in introductory courses, depending on the background in linear algebra that is assumed. Chapter 5 is intentionally designed to offer a number of plausible stopping points, depending on how much of this material the instructor wishes to cover. The first two sections can stand alone as an introduction to linear systems without the use of linear algebra and matrices. The last six sections of Chapter 5 employ the notation and terminology (although not so much of the theory) of elementary linear algebra. For ready reference, we have included in Section 5.3 a complete and self-contained account of the needed notation and terminology of determinants, matrices, and vectors.

The remainder of Chapter 5 has been substantially rewritten for this edition. Section 5.4 introduces the eigenvalue method for homogeneous first order linear systems, and includes applications of the case of simple (distinct) eigenvalues; discussion of multiple eigenvalues is deferred to Section 5.6. Section 5.5 (optional) applies the eigenvalue method to the second order linear systems that are typical of mechanical models, and provides a sample of the more technical applications of eigenvalues to physics and engineering problems. Sections 5.7 and 5.8 deal with nonhomogeneous linear systems and matrix exponentials, respectively.

Many instructors will choose to proceed directly from Chapter 5 to the study of nonlinear systems and stability in Chapter 7. This chapter is a considerable expansion of the corresponding chapter in the first two editions. We believe the importance of qualitative analysis of differential equations for elementary students is increasing, and therefore have made a special effort to make this material accessible to these students. The two initial sections of Chapter 7 provide a low-key introduction to stability and phase plane concepts. Sections 7.4 and 7.5 exhibit applications of stability to nonlinear ecological and mechanical systems, respectively.

Section 7.6 on chaos and bifurcation is new to this edition. It presents an elementary introduction to such contemporary topics as period-doubling toward chaos in biological and mechanical systems, the pitchfork diagram, and the Lorenz strange attractor.

Numerical Methods and Computing

In Chapter 6 on numerical methods, the perspective in which we view the subject is shaped by the wide availability of microcomputers on most campuses. With ready accessibility to substantial computing power, students can now envision the numerical approximation of solutions—and the graphical presentation of these approximate solutions—as routine and commonplace matters.

Our viewpoint in Chapter 6 is that understanding and appreciation of numerical algorithms is deepened by discussion of their computer implementations. We have included illustrative BASIC programs because no flowchart has the convincing tangibility of a program that actually runs (and produces the results claimed).

While serious scientific programming more often employs FORTRAN or Pascal, we feel that BASIC is best for elementary textbook exposition in mathematics—in BASIC we could include simple programs that without extensive discussion are intelligible and informative to students with little or no programming experience. With the basic understanding of numerical algorithms that these programs promote, the student is well prepared to use any of the DE software packages that now are readily available, or one of the general scientific computing environments, such as *Mathematica* or MATLAB.

In another vein, it is pointed out in the Chapter 1 summary that much of the numerical work in Chapter 6 can be covered at any point in the course subsequent to Chapter 1. In particular, instructors who are experimenting with the use of computers in teaching differential equations may wish to cover numerical methods earlier than has been the custom in the past.

Problems and Solutions

Probably in no other mathematics course beyond calculus are the exercises and problem sets so crucial to student learning as in the introductory differential equations course. We therefore devoted great effort to the development and selection of the approximately 1550 problems in this book. Each section contains more computational problems ("solve the following equations," and so on) than any class will ordinarily use, plus an ample number of applied problems. The answer section includes the answers to most odd-numbered problems and to some of the even-numbered ones.

In addition there is a Solutions Manual (of more than 380 pages) that accompanies this book. It includes answers to all of the problems, together with complete or partial solutions to most of the problems in the text that are not so routine that an answer alone suffices.

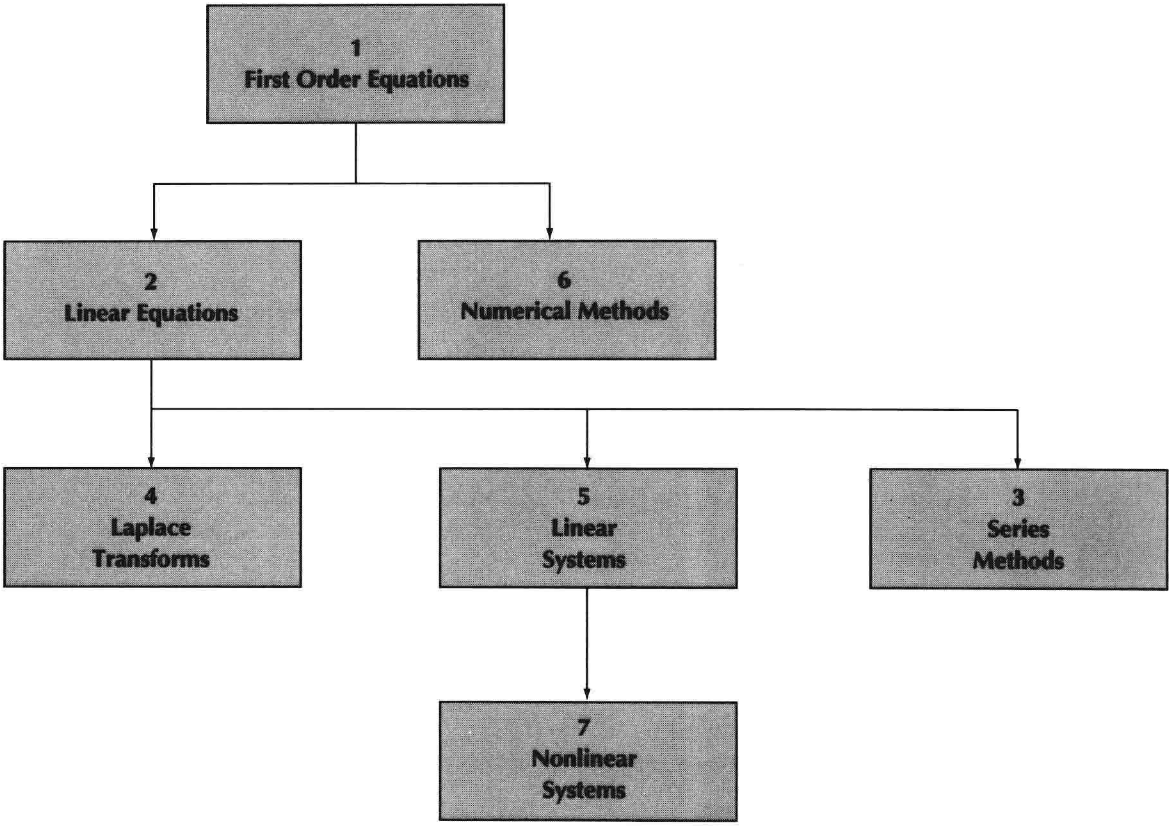
Acknowledgments

In preparing this revision and the earlier editions we profited greatly from the advice and assistance of the following very able reviewers. This edition was reviewed by: Donald Hartig, California Polytechnic State University; Frank G. Hagin, Colorado School of Mines; and Allan M. Krall, The Pennsylvania State University. Previous editions were reviewed by: Bruce Conrad, Temple University; W. Dan Curtis, Kansas State University; James W. Cushing, University of Arizona; Gertrude Ehrlich, University of Maryland; George Feissner, State University of New York at Cortland; Juan A. Gatica, University of Iowa; Robert Glassey, Indiana University; James L. Heitsch, University of Illinois at Chicago; Terry Herdman, Virginia Polytechnic Institute and State University; S. F. Neustadter, San Francisco State University; Anthony Peressini, University of Illinois; Thomas Rousseau, Siena College; William Rundell, Texas A & M University; and Erich Zauderer, Polytechnic Institute of New York. We

thank also our editor, Steven Conmy, for his enthusiastic and efficient coordination of the whole process. Once again, we are unable to express adequately our debts to Alice F. Edwards and Carol W. Penney for their continued assistance, encouragement, support, and patience.

C.H.E., Jr.
D.E.P.

DEPENDENCE OF CHAPTERS



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1

First Order Differential Equations

1.1 Introduction

The laws of the universe are written largely in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are best described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function f may be regarded as the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation involving an unknown function and one or more of its derivatives is called a **differential equation.**

EXAMPLE 1 The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x, together with the first two derivatives y' and y'' of y.

The study of differential equations has two principal goals:

1. To discover the differential equation that describes a specified physical situation;
2. To find the appropriate solution of that equation.

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation if possible.

EXAMPLE 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C(2xe^{x^2}) = (2x)(Ce^{x^2}) = 2xy.$$

Thus every function $y(x)$ of the form in (1) is a solution of the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all x. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the "arbitrary constant" C. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1).

Differential Equations and Mathematical Models

The following three examples illustrate the process of translating scientific laws and principles into differential equations, by interpreting rates of change as derivatives. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

EXAMPLE 3 Newton's law of cooling may be stated this way: The *time rate of change* (the rate of change with respect to time t) of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium. That is,

$$\frac{dT}{dt} = k(A - T), \tag{3}$$

where k is a positive constant. Observe that if $T > A$, then $dT/dt < 0$, so the temperature $T(t)$ is a decreasing function of t and the body is cooling. On the other hand, if $T < A$, then $dT/dt > 0$, so that T is increasing.

Thus the physical law is translated into a differential equation. If we are given the values of k and A, we hope to find an explicit formula for $T(t)$, and then—with the aid of this formula—we can predict the future temperature of the body.

EXAMPLE 4 The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{4}$$

where k is the constant of proportionality.

EXAMPLE 5 Torricelli's law implies that the *time rate of change* of the volume V of water in a draining tank is proportional to the square root of the depth y of the water in the tank:

$$\frac{dV}{dt} = -k\sqrt{y} \tag{5}$$