

Introduction to

PROBABILITY MODELS

EIGHTH EDITION



SHELDON M. ROSS

Introduction to Probability Models

Eighth Edition

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Preface



This text is intended as an introduction to elementary probability theory and stochastic processes. It is particularly well suited for those wanting to see how probability theory can be applied to the study of phenomena in fields such as engineering, computer science, management science, the physical and social sciences, and operations research.

It is generally felt that there are two approaches to the study of probability theory. One approach is heuristic and nonrigorous and attempts to develop in the student an intuitive feel for the subject which enables him or her to “think probabilistically.” The other approach attempts a rigorous development of probability by using the tools of measure theory. It is the first approach that is employed in this text. However, because it is extremely important in both understanding and applying probability theory to be able to “think probabilistically,” this text should also be useful to students interested primarily in the second approach.

New to This Edition

The eighth edition contains five new sections.

- Section 3.6.4 presents an elementary approach, using only conditional expectation, for computing the expected time until a sequence of independent and identically distributed random variables produce a specified pattern.
- Section 3.6.5 derives an identity involving compound Poisson random variables and then uses it to obtain an elegant recursive formula for the probabilities of compound Poisson random variables whose incremental increases are nonnegative and integer valued.

- Section 5.4.3 is concerned with a conditional Poisson process, a type of process that is widely applicable in the risk industries.
- Section 7.10 presents a derivation of and a new characterization for the classical insurance ruin probability.
- Section 11.8 presents a simulation procedure known as coupling from the past; its use enables one to exactly generate the value of a random variable whose distribution is that of the stationary distribution of a given Markov chain, even in cases where the stationary distribution cannot itself be explicitly determined.

There are also new Examples and Exercises in almost all chapters. Among the more significant are

- Examples 3.19, 3.28, 5.4 and 5.19, relating to insurance;
- Example 2.47 on the Poisson paradigm;
- Examples 4.7 and 4.23 on the Bonus-Malus system for setting automobile insurance premiums;
- Example 4.22, which shows how to obtain the expected time until a specified pattern appears in a sequence of Markov chain generated data;
- Example 5.1, which illustrates the connection between the total expected discounted return and the total expected (undiscounted) return earned by an exponentially distributed random time;
- Examples 11.19 and 11.20, which further indicate the use of variance reduction in obtaining efficient simulation estimators.

Course

Ideally, this text would be used in a one-year course in probability models. Other possible courses would be a one-semester course in introductory probability theory (involving Chapters 1–3 and parts of others) or a course in elementary stochastic processes. The textbook is designed to be flexible enough to be used in a variety of possible courses. For example, I have used Chapters 5 and 8, with smatterings from Chapters 4 and 6, as the basis of an introductory course in queueing theory.

Examples and Exercises

Many examples are worked out throughout the text, and there are also a large number of exercises to be solved by students. More than 100 of these exercises have been starred and their solutions provided at the end of the text. These starred

problems can be used for independent study and test preparation. An Instructor's Manual, containing solutions to all exercises, is available free to instructors who adopt the book for class.

Organization

Chapters 1 and 2 deal with basic ideas of probability theory. In Chapter 1 an axiomatic framework is presented, while in Chapter 2 the important concept of a random variable is introduced. Section 2.6.1 gives a simple derivation of the joint distribution of the sample mean and sample variance of a normal data sample.

Chapter 3 is concerned with the subject matter of conditional probability and conditional expectation. "Conditioning" is one of the key tools of probability theory, and it is stressed throughout the book. When properly used, conditioning often enables us to easily solve problems that at first glance seem quite difficult. The final section of this chapter presents applications to (1) a computer list problem, (2) a random graph, and (3) the Polya urn model and its relation to the Bose-Einstein distribution. Section 3.6.6 presents k -record values and the surprising Ignatov's theorem.

In Chapter 4 we come into contact with our first random, or stochastic, process, known as a Markov chain, which is widely applicable to the study of many real-world phenomena. Applications to genetics and production processes are presented. The concept of time reversibility is introduced and its usefulness illustrated. Section 4.5.3 presents an analysis, based on random walk theory, of a probabilistic algorithm for the satisfiability problem. Section 4.6 deals with the mean times spent in transient states by a Markov chain. Section 4.9 introduces Markov chain Monte Carlo methods. In the final section we consider a model for optimally making decisions known as a Markovian decision process.

In Chapter 5 we are concerned with a type of stochastic process known as a counting process. In particular, we study a kind of counting process known as a Poisson process. The intimate relationship between this process and the exponential distribution is discussed. New derivations for the Poisson and nonhomogeneous Poisson processes are discussed. Examples relating to analyzing greedy algorithms, minimizing highway encounters, collecting coupons, and tracking the AIDS virus, as well as material on compound Poisson processes are included in this chapter. Section 5.2.4 gives a simple derivation of the convolution of exponential random variables.

Chapter 6 considers Markov chains in continuous time with an emphasis on birth and death models. Time reversibility is shown to be a useful concept, as it is in the study of discrete-time Markov chains. Section 6.7 presents the computationally important technique of uniformization.

Chapter 7, the renewal theory chapter, is concerned with a type of counting process more general than the Poisson. By making use of renewal reward processes,

limiting results are obtained and applied to various fields. Section 7.9 presents new results concerning the distribution of time until a certain pattern occurs when a sequence of independent and identically distributed random variables is observed. In Section 7.9.1, we show how renewal theory can be used to derive both the mean and the variance of the length of time until a specified pattern appears, as well as the mean time until one of a finite number of specified patterns appears. In Section 7.9.2, we suppose that the random variables are equally likely to take on any of m possible values, and compute an expression for the mean time until a run of m distinct values occurs. In Section 7.9.3, we suppose the random variables are continuous and derive an expression for the mean time until a run of m consecutive increasing values occurs.

Chapter 8 deals with queueing, or waiting line, theory. After some preliminaries dealing with basic cost identities and types of limiting probabilities, we consider exponential queueing models and show how such models can be analyzed. Included in the models we study is the important class known as a network of queues. We then study models in which some of the distributions are allowed to be arbitrary. Included, are subsection (8.6.3) dealing with an optimization problem concerning a single server, general service time queue; and section (8.8) concerned with a single server, general service time queue in which the arrival source is a finite number of potential users.

Chapter 9 is concerned with reliability theory. This chapter will probably be of greatest interest to the engineer and operations researcher. Section 9.6.1 illustrates a method for determining an upper bound for the expected life of a parallel system of not necessarily independent components and (9.7.1) analyzing a series structure reliability model in which components enter a state of suspended animation when one of their cohorts fails.

Chapter 10 is concerned with Brownian motion and its applications. The theory of options pricing is discussed. Also, the arbitrage theorem is presented and its relationship to the duality theorem of linear program is indicated. We show how the arbitrage theorem leads to the Black-Scholes option pricing formula.

Chapter 11 deals with simulation, a powerful tool for analyzing stochastic models that are analytically intractable. Methods for generating the values of arbitrarily distributed random variables are discussed, as are variance reduction methods for increasing the efficiency of the simulation. Section 11.6.4 introduces the important simulation technique of importance sampling, and indicates the usefulness of tilted distributions when applying this method.

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Contents



Preface xiii

1. Introduction to Probability Theory 1

- 1.1. Introduction 1
- 1.2. Sample Space and Events 1
- 1.3. Probabilities Defined on Events 4
- 1.4. Conditional Probabilities 7
- 1.5. Independent Events 10
- 1.6. Bayes' Formula 12
 - Exercises 15
 - References 21

2. Random Variables 23

- 2.1. Random Variables 23
- 2.2. Discrete Random Variables 27
 - 2.2.1. The Bernoulli Random Variable 28
 - 2.2.2. The Binomial Random Variable 29
 - 2.2.3. The Geometric Random Variable 31
 - 2.2.4. The Poisson Random Variable 32
- 2.3. Continuous Random Variables 34
 - 2.3.1. The Uniform Random Variable 35
 - 2.3.2. Exponential Random Variables 36
 - 2.3.3. Gamma Random Variables 37
 - 2.3.4. Normal Random Variables 37

- 2.4. Expectation of a Random Variable 38
 - 2.4.1. The Discrete Case 38
 - 2.4.2. The Continuous Case 41
 - 2.4.3. Expectation of a Function of a Random Variable 43
- 2.5. Jointly Distributed Random Variables 47
 - 2.5.1. Joint Distribution Functions 47
 - 2.5.2. Independent Random Variables 51
 - 2.5.3. Covariance and Variance of Sums of Random Variables 53
 - 2.5.4. Joint Probability Distribution of Functions of Random Variables 61
- 2.6. Moment Generating Functions 64
 - 2.6.1. The Joint Distribution of the Sample Mean and Sample Variance from a Normal Population 74
- 2.7. Limit Theorems 77
- 2.8. Stochastic Processes 83
 - Exercises 85
 - References 96

3. Conditional Probability and Conditional Expectation 97

- 3.1. Introduction 97
- 3.2. The Discrete Case 97
- 3.3. The Continuous Case 102
- 3.4. Computing Expectations by Conditioning 105
 - 3.4.1. Computing Variances by Conditioning 116
- 3.5. Computing Probabilities by Conditioning 119
- 3.6. Some Applications 136
 - 3.6.1. A List Model 136
 - 3.6.2. A Random Graph 138
 - 3.6.3. Uniform Priors, Polya's Urn Model, and Bose–Einstein Statistics 146
 - 3.6.4. Mean Time for Patterns 150
 - 3.6.5. A Compound Poisson Identity 154
 - 3.6.6. The k -Record Values of Discrete Random Variables 158
- Exercises 161

4. Markov Chains 181

- 4.1. Introduction 181
- 4.2. Chapman–Kolmogorov Equations 185
- 4.3. Classification of States 189

- 4.4. Limiting Probabilities 200
- 4.5. Some Applications 213
 - 4.5.1. The Gambler's Ruin Problem 213
 - 4.5.2. A Model for Algorithmic Efficiency 217
 - 4.5.3. Using a Random Walk to Analyze a Probabilistic Algorithm for the Satisfiability Problem 220
- 4.6. Mean Time Spent in Transient States 226
- 4.7. Branching Processes 228
- 4.8. Time Reversible Markov Chains 232
- 4.9. Markov Chain Monte Carlo Methods 243
- 4.10. Markov Decision Processes 248
 - Exercises 252
 - References 268

5. The Exponential Distribution and the Poisson Process 269

- 5.1. Introduction 269
- 5.2. The Exponential Distribution 270
 - 5.2.1. Definition 270
 - 5.2.2. Properties of the Exponential Distribution 272
 - 5.2.3. Further Properties of the Exponential Distribution 279
 - 5.2.4. Convolutions of Exponential Random Variables 284
- 5.3. The Poisson Process 288
 - 5.3.1. Counting Processes 288
 - 5.3.2. Definition of the Poisson Process 289
 - 5.3.3. Interarrival and Waiting Time Distributions 293
 - 5.3.4. Further Properties of Poisson Processes 295
 - 5.3.5. Conditional Distribution of the Arrival Times 301
 - 5.3.6. Estimating Software Reliability 313
- 5.4. Generalizations of the Poisson Process 316
 - 5.4.1. Nonhomogeneous Poisson Process 316
 - 5.4.2. Compound Poisson Process 321
 - 5.4.3. Conditional or Mixed Poisson Processes 327
 - Exercises 330
 - References 348

6. Continuous-Time Markov Chains 349

- 6.1. Introduction 349
- 6.2. Continuous-Time Markov Chains 350
- 6.3. Birth and Death Processes 352

- 6.4. The Transition Probability Function $P_{ij}(t)$ 359
- 6.5. Limiting Probabilities 368
- 6.6. Time Reversibility 376
- 6.7. Uniformization 384
- 6.8. Computing the Transition Probabilities 388
 - Exercises 390
 - References 399

7. Renewal Theory and Its Applications 401

- 7.1. Introduction 401
- 7.2. Distribution of $N(t)$ 403
- 7.3. Limit Theorems and Their Applications 407
- 7.4. Renewal Reward Processes 416
- 7.5. Regenerative Processes 425
 - 7.5.1. Alternating Renewal Processes 428
- 7.6. Semi-Markov Processes 434
- 7.7. The Inspection Paradox 437
- 7.8. Computing the Renewal Function 440
- 7.9. Applications to Patterns 443
 - 7.9.1. Patterns of Discrete Random Variables 443
 - 7.9.2. The Expected Time to a Maximal Run of Distinct Values 451
 - 7.9.3. Increasing Runs of Continuous Random Variables 453
- 7.10. The Insurance Ruin Problem 455
 - Exercises 460
 - References 472

8. Queueing Theory 475

- 8.1. Introduction 475
- 8.2. Preliminaries 476
 - 8.2.1. Cost Equations 477
 - 8.2.2. Steady-State Probabilities 478
- 8.3. Exponential Models 480
 - 8.3.1. A Single-Server Exponential Queueing System 480
 - 8.3.2. A Single-Server Exponential Queueing System Having Finite Capacity 487
 - 8.3.3. A Shoeshine Shop 490
 - 8.3.4. A Queueing System with Bulk Service 493
- 8.4. Network of Queues 496
 - 8.4.1. Open Systems 496
 - 8.4.2. Closed Systems 501

- 8.5. The System $M/G/1$ 507
 - 8.5.1. Preliminaries: Work and Another Cost Identity 507
 - 8.5.2. Application of Work to $M/G/1$ 508
 - 8.5.3. Busy Periods 509
- 8.6. Variations on the $M/G/1$ 510
 - 8.6.1. The $M/G/1$ with Random-Sized Batch Arrivals 510
 - 8.6.2. Priority Queues 512
 - 8.6.3. An $M/G/1$ Optimization Example 515
- 8.7. The Model $G/M/1$ 519
 - 8.7.1. The $G/M/1$ Busy and Idle Periods 524
- 8.8. A Finite Source Model 525
- 8.9. Multiserver Queues 528
 - 8.9.1. Erlang's Loss System 529
 - 8.9.2. The $M/M/k$ Queue 530
 - 8.9.3. The $G/M/k$ Queue 530
 - 8.9.4. The $M/G/k$ Queue 532
- Exercises 534
- References 546

9. Reliability Theory 547

- 9.1. Introduction 547
- 9.2. Structure Functions 547
 - 9.2.1. Minimal Path and Minimal Cut Sets 550
- 9.3. Reliability of Systems of Independent Components 554
- 9.4. Bounds on the Reliability Function 559
 - 9.4.1. Method of Inclusion and Exclusion 560
 - 9.4.2. Second Method for Obtaining Bounds on $r(\mathbf{p})$ 569
- 9.5. System Life as a Function of Component Lives 571
- 9.6. Expected System Lifetime 580
 - 9.6.1. An Upper Bound on the Expected Life of a Parallel System 584
- 9.7. Systems with Repair 586
 - 9.7.1. A Series Model with Suspended Animation 591
- Exercises 593
- References 600

10. Brownian Motion and Stationary Processes 601

- 10.1. Brownian Motion 601
- 10.2. Hitting Times, Maximum Variable, and the Gambler's Ruin Problem 605

- 10.3. Variations on Brownian Motion 607
 - 10.3.1. Brownian Motion with Drift 607
 - 10.3.2. Geometric Brownian Motion 607
- 10.4. Pricing Stock Options 608
 - 10.4.1. An Example in Options Pricing 608
 - 10.4.2. The Arbitrage Theorem 611
 - 10.4.3. The Black–Scholes Option Pricing Formula 614
- 10.5. White Noise 620
- 10.6. Gaussian Processes 622
- 10.7. Stationary and Weakly Stationary Processes 625
- 10.8. Harmonic Analysis of Weakly Stationary Processes 630
 - Exercises 633
 - References 638

11. Simulation 639

- 11.1. Introduction 639
- 11.2. General Techniques for Simulating Continuous Random Variables 644
 - 11.2.1. The Inverse Transformation Method 644
 - 11.2.2. The Rejection Method 645
 - 11.2.3. The Hazard Rate Method 649
- 11.3. Special Techniques for Simulating Continuous Random Variables 653
 - 11.3.1. The Normal Distribution 653
 - 11.3.2. The Gamma Distribution 656
 - 11.3.3. The Chi-Squared Distribution 657
 - 11.3.4. The Beta (n, m) Distribution 657
 - 11.3.5. The Exponential Distribution—The Von Neumann Algorithm 658
- 11.4. Simulating from Discrete Distributions 661
 - 11.4.1. The Alias Method 664
- 11.5. Stochastic Processes 668
 - 11.5.1. Simulating a Nonhomogeneous Poisson Process 669
 - 11.5.2. Simulating a Two-Dimensional Poisson Process 676
- 11.6. Variance Reduction Techniques 679
 - 11.6.1. Use of Antithetic Variables 680
 - 11.6.2. Variance Reduction by Conditioning 684
 - 11.6.3. Control Variates 688
 - 11.6.4. Importance Sampling 690
- 11.7. Determining the Number of Runs 696

11.8. Coupling from the Past	696
Exercises	699
References	707

Appendix: Solutions to Starred Exercises 709

Index 749

1

Introduction to Probability Theory



1.1. Introduction

Any realistic model of a real-world phenomenon must take into account the possibility of randomness. That is, more often than not, the quantities we are interested in will not be predictable in advance but, rather, will exhibit an inherent variation that should be taken into account by the model. This is usually accomplished by allowing the model to be probabilistic in nature. Such a model is, naturally enough, referred to as a probability model.

The majority of the chapters of this book will be concerned with different probability models of natural phenomena. Clearly, in order to master both the “model building” and the subsequent analysis of these models, we must have a certain knowledge of basic probability theory. The remainder of this chapter, as well as the next two chapters, will be concerned with a study of this subject.

1.2. Sample Space and Events

Suppose that we are about to perform an experiment whose outcome is not predictable in advance. However, while the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by S .

Some examples are the following.

1. If the experiment consists of the flipping of a coin, then

$$S = \{H, T\}$$

where H means that the outcome of the toss is a head and T that it is a tail.