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INFORMATION-BASED COMPLEXITY

Joseph F. Traub,
G. W. Wasilkowski,
and H. Wozniakowski

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J. F. Traub
Department of Computer Science
Columbia University
New York, New York

G. W. Wasilkowski
Department of Computer Science
University of Kentucky
Lexington, Kentucky

H. Woźniakowski
Institute of Informatics
University of Warsaw
Warsaw, Poland
and
Department of Computer Science
Columbia University
New York, New York

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Section 4 of Chapter 5 was written by **T. Boult**, Department of Computer Science, Columbia University.

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Information-Based Complexity

This is a volume in
COMPUTER SCIENCE AND SCIENTIFIC COMPUTING
Werner Rheinboldt and Daniel Siewiorek, editors

To our children:

Claudia and Hillary Traub,

Bartek and Kubuś Wasilkowski,

Artur Woźniakowski,

and to my mother:

Jadwiga Wasilkowska.

Preface

The purpose of this book is to provide a comprehensive treatment of information-based complexity. We present theory and applications. In addition to integrating the work of many researchers, new results are also developed.

In two earlier books, we analyzed the worst case setting. Here we study the worst case, average case, probabilistic, random, and asymptotic settings. The effect of noisy information is briefly discussed. Some open problems are also indicated.

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J. F. Traub
G. W. Wasilkowski
H. Woźniakowski

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Chapter 1

Overview

The *computational complexity* of a problem is its intrinsic difficulty as measured by the minimal computational resources, such as time or memory, required for its solution. Equivalently, the computational complexity is the minimal cost among all algorithms that solve the problem. Such a minimal cost algorithm is said to be *optimal.* The computational complexity is a *problem invariant*; it is independent of any particular algorithm. The notion of an invariant is important in many scientific fields. We believe that computational complexity is a fundamental invariant of computer science.

Computational complexity sets intrinsic limits on which problems can be solved. Problems that cannot be solved because limitations dictate that the requisite computational resources can never be achieved are said to be *intractable.* Having provided a benchmark for the intrinsic difficulty of a problem, one may compare its computational complexity with the cost of any algorithm that solves the problem to tell how well the given algorithm measures up.

The reader will note from our usage above that the term *computational complexity* serves double duty, both as the name of a scientific field, as well as a crucial problem invariant within that field.

A central notion in the theory which is developed in this book is *information.* We do not mean information in the sense of Claude Shannon and information theory. For present purposes, information is what we know about the problem to be solved. Information will be defined in the general formulation of Chapter 3.

Most problems arising in the sciences or engineering have the characteristic that information relevant to their solution is either partial or noisy. For such problems, only approximate solutions are possible.

As a digital computer can only manipulate a finite set of numbers, any problem whose domain of possible problem elements is infinite will necessarily have only partial information. In particular, this is true of continuous problems defined on an infinite dimensional function space.

We emphasize that information-based complexity is not restricted to infinite dimensional problems. For instance, it can be used to study finite dimensional problems with complete but noisy information. Also, it can be used if the complexity of computing an exact solution is prohibitively large and one is willing to settle for an approximate solution to reduce the complexity.

As a simple example of partial and noisy information consider the computation of a definite integral. For most integrands we cannot compute the integral utilizing the fundamental theorem of the calculus since there is no closed form expression for the antiderivative. We have to approximate the integral numerically. Usually, the integrand is evaluated at a finite number of points. The information is the values of the integrand at these points. In general, an infinite number of integrands have the same values at these points, and therefore the information is partial. In addition, there will be round-off error in evaluating the integrand, and so the information is noisy. The integral is estimated by combining the integrand values. Since the information we are using does not uniquely identify the integrand, we can compute only an approximate solution. There is intrinsic uncertainty in the answer.

For problems arising in science and engineering the information has another characteristic; it is *priced*. For instance, in the integration example we should be charged for the evaluations of the integrand.

The branch of computational complexity that deals with the intrinsic difficulty of the approximate solution of problems for which the information is partial, noisy, and priced is called *information-based complexity*.

Problems with partial, noisy, and priced information arise in many areas. These include economics, numerical analysis, physics, human and robotic vision, scientific and engineering computation, geophysics, decision theory, signal processing, and control theory.

On the other hand, there are problems for which the information is *complete*, *exact*, and *free*. The branch of computational complexity that deals with such problems is known as *combinatorial complexity*. An example is provided by the traveling salesman problem. In this problem the information is a set of cities along with the distances between all pairs of cities. Since the information specifies the problem uniquely, the information is complete. Furthermore, this information is exact and free. These assumptions are typical for many other important problems, for example, for NP-complete problems.

In both combinatorial complexity and information-based complexity we seek to solve a problem using an algorithm with minimal cost. Since problems of information-based complexity can only be solved approximately, the notion of *error* is important. We require that the problem be solved with error no greater than a threshold ε. The *ε-complexity* is then defined as the minimal cost among all algorithms which solve the problem with error at most ε.

The cost and error of algorithms can be variously defined, leading to different settings. In the *worst case* setting, the cost and error are defined by their worst performance. In the *average case* setting, the cost and error are defined by their average performance. In the *probabilistic* setting, errors on sets of small measure are ignored. Mixed settings, where the cost is defined in one sense and the error in a different sense, are also of interest.

An error criterion must be specified. Absolute and relative criteria are among those studied. Since results are sensitive to the error criteria, this further enriches the subject.

The purpose of this book is to provide a comprehensive treatment of information-based complexity. We shall sometimes refer to two earlier books. Both of these deal with the worst case setting:

- *A General Theory of Optimal Algorithms*, J. F. Traub and H. Woźniakowski, Academic Press, 1980. We will refer to this book as GTOA.
- *Information, Uncertainty, Complexity*, J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski, Addison-Wesley, 1983. We will refer to this book as IUC.

We briefly indicate the contents of this book. In Chapter 2 we use the example of continuous binary search to illustrate the main issues and concepts of information-based complexity. Results are obtained for the major settings and error criteria. In the following chapter we present an abstract formulation of information-based complexity and rigorously define such concepts as information, ε-complexity, optimal information, and optimal algorithm. The major settings are also defined.

The theory of the worst case setting is given in Chapter 4. Twelve applications for the worst case setting are discussed in Chapter 5.

Chapters 6 and 7 develop theory and applications for the average case setting, while Chapter 8 deals with theory and applications for the probabilistic setting.

In Chapter 9 we take a different cut through the subject. We fix the problem and compare the ε-complexities for varying settings and error criteria. We do this for the integration and function approximation problems.

In Chapter 10 we study the asymptotic setting which is extensively used in numerical analysis. The objective in this setting is to achieve the best possible speed of convergence. Relations are obtained between the asymptotic and worst case settings, and between the asymptotic and average case settings.

Up to this point, the book is restricted to deterministic information and algorithms. In Chapter 11 we study random information and random algorithms.

Chapter 12 deals with the important but technically difficult case of noisy information. The worst case and average cases are discussed.

For the reader's benefit, concepts and results from functional analysis and measure theory used in this book are summarized in the appendix. The bibliography contains over 440 entries.

Typically, a chapter and its sections are followed by Notes and Remarks which include commentary on the text, extensions to the results, and bibliographical discussion. In Notes and Remarks we also give reference to papers with original results as well as papers which serve as a basis of the chapter or section. The lack of such a reference indicates that the analysis is new. Sections sometimes conclude with exercises for the reader.

We end this overview by describing our system for referring to material within the text. Theorems, equations, remarks, etc. are separately numbered for each section. A reference to material within the same section does not name the section. A reference to material within a different section of the same chapter names the section, and a reference to material within a different chapter names the chapter and the section.

Notes and Remarks as well as Exercises are also separately numbered for each section. For instance, **NR 5.4:3** denotes the third entry in Notes and Remarks in Section 5.4 of a given chapter, and **E 2:1** denotes the first exercise in Section 2 of a given chapter.

Notes and Remarks

NR 1:1 For work on information-based complexity up to 1980 see GTOA. It deals with partial information and the worst case setting. A brief history and an annotated bibliography of 325 books and papers is included.

IUC deals with partial or noisy information in the worst case setting. It utilizes a more general framework than that used in most of this book, see Remark 2.1 of Chapter 3.

A number of surveys have been written for certain audiences or emphasizing particular viewpoints. These include Traub and Woźniakowski [84a], Wasilkowski [85], Woźniakowski [85, 86a], Packel and Traub [87], Packel, Traub, and Woźniakowski [87], and Packel and Woźniakowski [87].