

**Computer Science  
and Scientific Computing**

---

**INFORMATION-BASED  
COMPLEXITY**

**Joseph F. Traub,  
G. W. Wasilkowski,  
and H. Wozniakowski**

# Information-Based Complexity

**J. F. Traub**

*Department of Computer Science  
Columbia University  
New York, New York*

**G. W. Wasilkowski**

*Department of Computer Science  
University of Kentucky  
Lexington, Kentucky*

**H. Woźniakowski**

*Institute of Informatics  
University of Warsaw  
Warsaw, Poland*

and

*Department of Computer Science  
Columbia University  
New York, New York*



ACADEMIC PRESS, INC.

*Harcourt Brace Jovanovich, Publishers*

Boston San Diego New York

Berkeley London Sydney

Tokyo Toronto

Copyright © 1988 by Academic Press, Inc.

All right reserved.

No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without permission in writing from the publisher.

ACADEMIC PRESS, INC.

1250 Sixth Avenue, San Diego, CA 92101

*United Kingdom Edition published by*

ACADEMIC PRESS INC. (LONDON) LTD.

24-28 Oval Road, London NW1 7DX

Sections 5-7 of Chapter 5, Section 4 of Chapter 7, and part of Section 7 of Chapter 8 were written by **A. G. Werschulz**, Division of Science and Mathematics, Fordham University, and Department of Computer Science, Columbia University.

Section 4 of Chapter 5 was written by **T. Boulton**, Department of Computer Science, Columbia University.

#### Library of Congress Cataloging-in-Publication Data

Traub, J. F. (Joseph Frederick), 1932-

Information-based complexity / J. F. Traub, G. W. Wasilkowski, H. Wozniakowski.

p. cm. - (Computer science and scientific computing)

Bibliography: p.

Includes indexes.

ISBN 0-12-697545-0

1. Computational complexity. I. Wasilkowski, G. W.

II. Wozniakowski, H. III. Title. IV. Series.

QA267.T73 1988

511.3-dc19

87-28918

CIP

88 89 90 91 9 8 7 6 5 4 3 2 1

Printed in the United States of America

## **Information-Based Complexity**

This is a volume in  
**COMPUTER SCIENCE AND SCIENTIFIC COMPUTING**  
Werner Rheinboldt and Daniel Siewiorek, editors

To our children:

Claudia and Hillary Traub,  
Bartek and Kubuś Wasilkowski,  
Artur Woźniakowski,

and to my mother:

Jadwiga Wasilkowska.

---

## Preface

The purpose of this book is to provide a comprehensive treatment of information-based complexity. We present theory and applications. In addition to integrating the work of many researchers, new results are also developed.

In two earlier books, we analyzed the worst case setting. Here we study the worst case, average case, probabilistic, random, and asymptotic settings. The effect of noisy information is briefly discussed. Some open problems are also indicated.

We wish to acknowledge many debts. Our special thanks are to A. G. Werschulz who authored several sections, as indicated on the front page, suggested many improvements to the whole book and was of invaluable help in preparing the  $\text{\TeX}$  version of the book and in preparing the two indices. We thank T. Boult who wrote the computer vision section. We are pleased to thank S. Kwapien for useful remarks and guidance concerning measure theory.

We appreciate valuable suggestions from M. A. Kowalski and K. Sikorski, and comments from M. A. Kon, D. Lee, E. W. Packel, and L. Plaskota. We also thank T. Orowan who superbly typed a number of the chapters in  $\text{\TeX}$ .

We are pleased to thank the National Science Foundation (Grant ICT-85-17289 and DCR-86-03674) and the Advanced Research Projects Agency (Contract N00039-84-C-0165) for supporting the work reported here.

J. F. Traub  
G. W. Wasilkowski  
H. Woźniakowski

---

## Table of Contents

Preface . . . . .	xiii
Chapter 1. Overview . . . . .	1
Chapter 2. Example: Continuous Binary Search . . . . .	5
1. Introduction . . . . .	5
1.1. Problem Formulation . . . . .	6
1.2. Information . . . . .	6
1.3. Model of Computation . . . . .	6
2. Complexity . . . . .	7
3. Worst Case Setting . . . . .	8
4. Average Case Setting . . . . .	10
5. Probabilistic Setting . . . . .	12
6. Relative Error . . . . .	13
6.1. Worst Case Setting . . . . .	14
6.2. Average Case Setting . . . . .	15
6.3. Probabilistic Setting . . . . .	17
7. Comparison of Complexity . . . . .	18
8. Mixed Settings . . . . .	19
9. Noisy Information . . . . .	20
10. Final Comments . . . . .	21
Chapter 3. General Formulation . . . . .	23
1. Introduction . . . . .	23



2. Formulation . . . . .	24
2.1. Problem Formulation . . . . .	24
2.2. Information . . . . .	27
2.3. Model of Computation . . . . .	30
2.4. How Can We Compute Approximations? . . . . .	33
3. Complexity in Three Settings . . . . .	35
4. Asymptotic Setting . . . . .	38
5. Randomization . . . . .	38
 Chapter 4. Worst Case Setting: Theory . . . . .	 41
1. Introduction . . . . .	41
2. Radius and Diameter of Information . . . . .	43
3. Algorithms . . . . .	48
3.1. Local and Global Errors . . . . .	48
3.2. Central Algorithms . . . . .	49
3.3. Interpolatory Algorithms . . . . .	51
4. Cardinality Number and Complexity . . . . .	53
5. Linear Problems . . . . .	55
5.1. Definition of Linear Problems . . . . .	56
5.2. Adaption versus Nonadaption . . . . .	57
5.3. Optimal Information . . . . .	65
5.4. Relations between $n$ th Minimal Radii and Gelfand $n$ -Widths . . . . .	70
5.5. Linear Algorithms . . . . .	75
5.6. Optimal Linear Algorithms and Linear Kolmogorov $n$ -Widths . . . . .	91
5.7. Spline Algorithms . . . . .	95
5.8. Complexity . . . . .	101
6. Different Error Criteria . . . . .	105
6.1. Relative Error . . . . .	105
6.2. Normalized Error . . . . .	109
6.3. Convex and Symmetric Error . . . . .	113
 Chapter 5. Worst Case Setting: Applications . . . . .	 117
1. Introduction . . . . .	117
2. Integration . . . . .	117
2.1. Smooth Periodic Functions . . . . .	119
2.2. Smooth Nonperiodic Functions . . . . .	123
2.3. Weighted Integration in a Reproducing Kernel Hilbert Space . . . . .	131
3. Function Approximation . . . . .	137
3.1. Smooth Periodic Functions . . . . .	138
3.2. Smooth Nonperiodic Functions: Hilbert Case . . . . .	144
3.3. Smooth Nonperiodic Functions: Banach Case . . . . .	147
4. Computer Vision . . . . .	148
5. Linear Partial Differential Equations . . . . .	153
6. Integral Equations . . . . .	167

7. Ill-Posed Problems . . . . .	173
8. Optimization . . . . .	177
9. Large Linear Systems . . . . .	178
10. Eigenvalue Problem . . . . .	183
11. Ordinary Differential Equations . . . . .	186
12. Nonlinear Equations . . . . .	188
13. Topological Degree . . . . .	193
 Chapter 6. Average Case Setting: Theory . . . . .	 195
1. Introduction . . . . .	195
2. Radius of Information . . . . .	197
3. Algorithms . . . . .	204
3.1. Local and Global Average Errors . . . . .	204
3.2. Central Algorithms . . . . .	206
4. Average Cardinality Number and Complexity . . . . .	213
5. Linear Problems . . . . .	215
5.1. Linear Problems in the Average Case Setting . . . . .	217
5.2. Gaussian Measures . . . . .	218
5.3. Central Algorithms . . . . .	222
5.4. Spline Algorithms . . . . .	226
5.5. Optimal Nonadaptive Information . . . . .	233
5.6. Adaptive Information . . . . .	236
5.6.1. Adaptive Information with Fixed Cardinality . . . . .	237
5.6.2. Adaptive Information with Varying Cardinality . . . . .	240
5.7. Complexity . . . . .	248
5.8. Linear Problems for Bounded Domains . . . . .	257
5.8.1. Average Radius of Information . . . . .	259
5.8.2. Proof of Theorem 5.8.1. . . . .	265
6. Different Error Criteria . . . . .	268
6.1. Relative Error . . . . .	268
6.2. Normalized Error . . . . .	279
6.3. General Error Functional . . . . .	281
6.4. Precision Error . . . . .	292
 Chapter 7. Average Case Setting : Applications . . . . .	 296
1. Introduction . . . . .	296
2. Integration . . . . .	297
2.1. Smooth Functions . . . . .	297
2.2. Weighted Integration in a Reproducing Kernel Hilbert Space . . . . .	302
3. Function Approximation . . . . .	309
3.1. Smooth Periodic Functions . . . . .	309
3.2. Smooth Nonperiodic Functions . . . . .	311
4. Ill-Posed Problems . . . . .	315

Chapter 8. Probabilistic Setting . . . . .	323
1. Introduction . . . . .	323
2. Relation to Average Case Setting . . . . .	326
3. Radius of Information . . . . .	327
4. Probabilistic Cardinality Number and Complexity . . . . .	328
5. Linear Problems . . . . .	329
5.1. Optimal Error Algorithms . . . . .	329
5.2. Estimates of the Radius of Information . . . . .	331
5.3. Optimal Information . . . . .	334
5.4. Complexity . . . . .	336
5.5. Linear Problems for Bounded Domains . . . . .	344
6. Different Error Criteria . . . . .	346
6.1. Relative Error . . . . .	346
6.2. Normalized Error . . . . .	356
6.3. General Error Functional . . . . .	358
7. Applications . . . . .	359
Chapter 9. Comparison between Different Settings . . . . .	364
1. Introduction . . . . .	364
2. Integration of Smooth Functions . . . . .	365
3. Integration of Smooth Periodic Functions . . . . .	368
4. Approximation of Smooth Periodic Functions . . . . .	370
5. Approximation of Smooth Nonperiodic Functions . . . . .	372
Chapter 10. Asymptotic Setting . . . . .	375
1. Introduction . . . . .	375
2. Asymptotic and Worst Case Settings: Linear Problems . . . . .	383
2.1. Optimal Algorithms . . . . .	383
2.2. Optimal Information . . . . .	389
2.3. Continuous Algorithms . . . . .	394
3. Asymptotic and Worst Case Settings: Nonlinear Problems . . . . .	395
3.1. Optimal Algorithms . . . . .	395
3.2. Optimal Information . . . . .	399
3.3. Applications . . . . .	400
4. Asymptotic and Average Case Settings . . . . .	403
4.1. Optimal Algorithms . . . . .	404
4.2. Rate of Convergence . . . . .	407
4.3. Optimal Information . . . . .	410

Chapter 11. Randomization . . . . .	413
1. Introduction . . . . .	413
2. Random Information and Random Algorithms . . . . .	414
3. Average Case Setting . . . . .	418
3.1. Linear Problems for the Whole Space . . . . .	419
3.2. Linear Problems for Bounded Domains . . . . .	421
4. Worst Case Setting . . . . .	422
4.1. Function Approximation and Other Problems . . . . .	423
4.1.1. Function Approximation . . . . .	425
4.1.2. Maximum and Extremal Points . . . . .	425
4.1.3. Function Inverse . . . . .	426
4.1.4. Topological Degree . . . . .	427
4.2. Integration . . . . .	429
4.3. Linear Problems with Unrestricted Information . . . . .	431
Chapter 12. Noisy Information . . . . .	434
1. Introduction . . . . .	434
2. Worst Case Setting with Deterministic Noise . . . . .	434
2.1. Basic Definitions . . . . .	434
2.2. Uniformly Bounded Noise . . . . .	436
3. Average Case Setting with Random Noise . . . . .	441
3.1. Basic Definitions . . . . .	441
3.2. Normally Distributed Noise . . . . .	443
3.3. Does Adaption Help? . . . . .	443
3.3.1. Examples . . . . .	444
3.3.2. Adaptive Choice of Observations Does Not Help . . . . .	445
4. Mixed Setting . . . . .	450
Appendix . . . . .	453
1. Functional Analysis . . . . .	453
1.1. Linear Spaces and Linear Operators . . . . .	453
1.2. Linear Independence, Dimension, and Linear Subspaces . . . . .	454
1.3. Norms and Continuous Linear Operators . . . . .	455
1.4. Banach Spaces . . . . .	456
1.5. Inner Products and Hilbert Spaces . . . . .	456
1.5.1. Inner Products . . . . .	456
1.5.2. Hilbert Spaces . . . . .	457
1.5.3. Separable Hilbert Spaces . . . . .	458
1.6. Bounded Operators on Hilbert Spaces . . . . .	458
1.6.1. Bounded Functionals and Riesz's Theorem . . . . .	458
1.6.2. Adjoint Operators . . . . .	458
1.6.3. Orthogonal and Projection Operators . . . . .	459
1.6.4. Spectrum . . . . .	460

2. Measure Theory . . . . . 461

2.1. Borel  $\sigma$ -Field, Measurable Sets and Functions . . . . . 461

2.2. Measures and Probability Measures . . . . . 461

2.3. Integrals . . . . . 462

2.4. Characteristic Functional . . . . . 464

2.5. Mean Element . . . . . 464

2.6. Covariance and Correlation Operators . . . . . 465

2.7. Induced and Conditional Measures . . . . . 465

2.8. Product Measures and Fubini's Theorem . . . . . 466

2.9. Gaussian Measures . . . . . 466

2.9.1. Measure of a Ball . . . . . 467

2.9.2. Induced and Conditional Measures . . . . . 471

Bibliography . . . . . 475

Author Index . . . . . 511

Subject Index . . . . . 517

## Chapter 1

---

### Overview

The *computational complexity* of a problem is its intrinsic difficulty as measured by the minimal computational resources, such as time or memory, required for its solution. Equivalently, the computational complexity is the minimal cost among all algorithms that solve the problem. Such a minimal cost algorithm is said to be *optimal*. The computational complexity is a *problem invariant*; it is independent of any particular algorithm. The notion of an invariant is important in many scientific fields. We believe that computational complexity is a fundamental invariant of computer science.

Computational complexity sets intrinsic limits on which problems can be solved. Problems that cannot be solved because limitations dictate that the requisite computational resources can never be achieved are said to be *intractable*. Having provided a benchmark for the intrinsic difficulty of a problem, one may compare its computational complexity with the cost of any algorithm that solves the problem to tell how well the given algorithm measures up.

The reader will note from our usage above that the term *computational complexity* serves double duty, both as the name of a scientific field, as well as a crucial problem invariant within that field.

A central notion in the theory which is developed in this book is *information*. We do not mean information in the sense of Claude Shannon and information theory. For present purposes, information is what we know about the problem to be solved. Information will be defined in the general formulation of Chapter 3.

Most problems arising in the sciences or engineering have the characteristic that information relevant to their solution is either partial or noisy. For such problems, only approximate solutions are possible.

As a digital computer can only manipulate a finite set of numbers, any problem whose domain of possible problem elements is infinite will necessarily have only partial information. In particular, this is true of continuous problems defined on an infinite dimensional function space.

We emphasize that information-based complexity is not restricted to infinite dimensional problems. For instance, it can be used to study finite dimensional problems with complete but noisy information. Also, it can be used if the complexity of computing an exact solution is prohibitively large and one is willing to settle for an approximate solution to reduce the complexity.

As a simple example of partial and noisy information consider the computation of a definite integral. For most integrands we cannot compute the integral utilizing the fundamental theorem of the calculus since there is no closed form expression for the antiderivative. We have to approximate the integral numerically. Usually, the integrand is evaluated at a finite number of points. The information is the values of the integrand at these points. In general, an infinite number of integrands have the same values at these points, and therefore the information is partial. In addition, there will be round-off error in evaluating the integrand, and so the information is noisy. The integral is estimated by combining the integrand values. Since the information we are using does not uniquely identify the integrand, we can compute only an approximate solution. There is intrinsic uncertainty in the answer.

For problems arising in science and engineering the information has another characteristic; it is *priced*. For instance, in the integration example we should be charged for the evaluations of the integrand.

The branch of computational complexity that deals with the intrinsic difficulty of the approximate solution of problems for which the information is partial, noisy, and priced is called *information-based complexity*.

Problems with partial, noisy, and priced information arise in many areas. These include economics, numerical analysis, physics, human and robotic vision, scientific and engineering computation, geophysics, decision theory, signal processing, and control theory.

On the other hand, there are problems for which the information is *complete*, *exact*, and *free*. The branch of computational complexity that deals with such problems is known as *combinatorial complexity*. An example is provided by the traveling salesman problem. In this problem the information is a set of cities along with the distances between all pairs of cities. Since the information specifies the problem uniquely, the information is complete. Furthermore, this information is exact and free. These assumptions are typical for many other important problems, for example, for NP-complete problems.

In both combinatorial complexity and information-based complexity we seek to solve a problem using an algorithm with minimal cost. Since problems of information-based complexity can only be solved approximately, the notion of *error* is important. We require that the problem be solved with error no greater than a threshold  $\varepsilon$ . The  $\varepsilon$ -*complexity* is then defined as the minimal cost among all algorithms which solve the problem with error at most  $\varepsilon$ .

The cost and error of algorithms can be variously defined, leading to different settings. In the *worst case* setting, the cost and error are defined by their worst performance. In the *average case* setting, the cost and error are defined by their average performance. In the *probabilistic* setting, errors on sets of small measure are ignored. Mixed settings, where the cost is defined in one sense and the error in a different sense, are also of interest.

An error criterion must be specified. Absolute and relative criteria are among those studied. Since results are sensitive to the error criteria, this further enriches the subject.

The purpose of this book is to provide a comprehensive treatment of information-based complexity. We shall sometimes refer to two earlier books. Both of these deal with the worst case setting:

- *A General Theory of Optimal Algorithms*, J. F. Traub and H. Woźniakowski, Academic Press, 1980. We will refer to this book as GTOA.
- *Information, Uncertainty, Complexity*, J. F. Traub, G. W. Wasilowski, and H. Woźniakowski, Addison-Wesley, 1983. We will refer to this book as IUC.

We briefly indicate the contents of this book. In Chapter 2 we use the example of continuous binary search to illustrate the main issues and concepts of information-based complexity. Results are obtained for the major settings and error criteria. In the following chapter we present an abstract formulation of information-based complexity and rigorously define such concepts as information,  $\varepsilon$ -complexity, optimal information, and optimal algorithm. The major settings are also defined.

The theory of the worst case setting is given in Chapter 4. Twelve applications for the worst case setting are discussed in Chapter 5.

Chapters 6 and 7 develop theory and applications for the average case setting, while Chapter 8 deals with theory and applications for the probabilistic setting.

In Chapter 9 we take a different cut through the subject. We fix the problem and compare the  $\varepsilon$ -complexities for varying settings and error criteria. We do this for the integration and function approximation problems.

In Chapter 10 we study the asymptotic setting which is extensively used



in numerical analysis. The objective in this setting is to achieve the best possible speed of convergence. Relations are obtained between the asymptotic and worst case settings, and between the asymptotic and average case settings.

Up to this point, the book is restricted to deterministic information and algorithms. In Chapter 11 we study random information and random algorithms.

Chapter 12 deals with the important but technically difficult case of noisy information. The worst case and average cases are discussed.

For the reader's benefit, concepts and results from functional analysis and measure theory used in this book are summarized in the appendix. The bibliography contains over 440 entries.

Typically, a chapter and its sections are followed by Notes and Remarks which include commentary on the text, extensions to the results, and bibliographical discussion. In Notes and Remarks we also give reference to papers with original results as well as papers which serve as a basis of the chapter or section. The lack of such a reference indicates that the analysis is new. Sections sometimes conclude with exercises for the reader.

We end this overview by describing our system for referring to material within the text. Theorems, equations, remarks, etc. are separately numbered for each section. A reference to material within the same section does not name the section. A reference to material within a different section of the same chapter names the section, and a reference to material within a different chapter names the chapter and the section.

Notes and Remarks as well as Exercises are also separately numbered for each section. For instance, **NR 5.4:3** denotes the third entry in Notes and Remarks in Section 5.4 of a given chapter, and **E 2:1** denotes the first exercise in Section 2 of a given chapter.

### Notes and Remarks

**NR 1:1** For work on information-based complexity up to 1980 see GTOA. It deals with partial information and the worst case setting. A brief history and an annotated bibliography of 325 books and papers is included.

IUC deals with partial or noisy information in the worst case setting. It utilizes a more general framework than that used in most of this book, see Remark 2.1 of Chapter 3.

A number of surveys have been written for certain audiences or emphasizing particular viewpoints. These include Traub and Woźniakowski [84a], Wasilkowski [85], Woźniakowski [85, 86a], Packel and Traub [87], Packel, Traub, and Woźniakowski [87], and Packel and Woźniakowski [87].