CONTEMPORARY MATHEMATICS

325

Combinatorial and Geometric Representation Theory

An International Conference on
Combinatorial and Geometric Representation Theory
October 22–26, 2001
Seoul National University, Seoul, Korea

Seok-Jin Kang
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This volume represents the proceedings of an international conference on Combinatorial and Geometric Representation Theory that was held at Seoul National University, Seoul, Korea, from October 22–26, 2001.

2000 *Mathematics Subject Classification.* Primary 05Exx, 14Lxx, 16Gxx, 17Bxx, 20Cxx, 20Gxx, 81Rxx.

Library of Congress Cataloging-in-Publication Data

Combinatorial and geometric representation theory : an international conference on combinatorial and geometric representation theory, October 22–26, 2001, Seoul National University, Seoul, Korea / Seok-Jin Kang, Kyu-Hwan Lee, editors.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 325)

Includes bibliographical references.

ISBN 0-8218-3212-3 (alk. paper)

1. Representations of groups—Congresses. 2. Representations of algebras—Congresses. 3. Combinatorial analysis—Congresses. 4. Geometry—Congresses. I. Kang, Seok-Jin. II. Lee, Kyu-Hwan, 1970– III. Contemporary mathematics (American Mathematical Society) ; v. 325.

QA176.C66 2001
512′.2—dc21 2002041753

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10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

Combinatorial and Geometric Representation Theory

Preface

This volume is the refereed proceedings of the international conference on "Combinatorial and Geometric Representation Theory" that was held at Seoul National University, Seoul, Korea, from October 22nd to October 26th, 2001.

In the area of representation theory, a wide variety of mathematical ideas has been combined together to provide new insights into the field, powerful methods of understanding the theory, and various applications to other branches of mathematics. Over the past two decades, there have been remarkable developments in representation theory based on combinatorial and geometric approaches.

The theme of this conference was to bring together various ideas from combinatorial and geometric aspects of representation theory and discuss the recent developments in this active field of research. We hope this conference served as a good opportunity to understand strong connections between combinatorics, geometry and representation theory.

We are very grateful to all the invited speakers and participants for their excellent lectures, contributed papers and great enthusiasm. We would also like to thank graduate students of Seoul National University for their assistance during the conference. Special thanks should be given to Professors Young-Hyun Cho, Myung-Hwan Kim and In-Sok Lee who served as members of the organizing committee of this conference.

This conference was supported by KOSEF Grant 98-0701-01-5-L. We greatly appreciate their financial and moral support. Finally, we would like to express our gratitude to all the referees for their invaluable help with the contributed papers.

Seok-Jin Kang
Kyu-Hwan Lee
Editors

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Contemporary Mathematics
Volume **325**, 2003

Twisted Verma modules and their quantized analogues

Henning Haahr Andersen

1. Introduction

In [**AL**] we studied twisted Verma modules for a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$. In fact, we gave three rather different constructions which we showed lead to the same modules. Here we shall briefly recall one of these approaches – the one based on Arkhipov's twisting functors [**Ar**]. We then demonstrate that this construction can also be used for the quantized enveloping algebra $U_q(\mathfrak{g})$.

In analogy with their classical counterparts the quantized twisted Verma modules belong to the category $\mathcal{O}_q$ for $U_q(\mathfrak{g})$ and have the same composition factors as the ordinary Verma modules for $U_q(\mathfrak{g})$. They also possess Jantzen type filtrations with corresponding sum formulae.

I would like to thank Catharina Stroppel and Niels Lauritzen for some very helpful comments.

2. The classical case

2.1. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$ and choose a set R^+ of positive roots in the root system R attached to $(\mathfrak{g}, \mathfrak{h})$. Then we have the usual triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$ with $\mathfrak{n}^+$ (respectively $\mathfrak{n}^-$) denoting the nilpotent subalgebra corresponding to the positive (respectively negative) roots.

We set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and write $U = U(\mathfrak{g})$ and $B = U(\mathfrak{b})$ for the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{b}$. Then the Verma module corresponding to $\lambda \in \mathfrak{h}^*$ is defined as

$$M(\lambda) = U \otimes_B \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional B-module obtained by composing λ with the projection $\mathfrak{b} \to \mathfrak{h}$.

Supported in part by the TMR programme "Algebraic Lie Representations" (ECM Network Contract No. ERB FMRX-CT 97/0100)

2.2. For each w in the Weyl group W we constructed in [**AL**] a twisted Verma module $M^w(\lambda)$. We shall now briefly recall one of the constructions of $M^w(\lambda)$.

First we associate to w a semiregular module S_w. It is defined by considering the subalgebra $\mathfrak{n}_w = \mathfrak{n}^- \cap w^{-1}(\mathfrak{n}^+)$ of $\mathfrak{n}^-$ and its corresponding enveloping algebra $N_w = U(\mathfrak{n}_w)$. The standard $\mathbb{Z}R$-grading on U and its associated $\mathbb{Z}$-grading $U = \bigoplus_{m\in\mathbb{Z}} U_m$ obtained via the natural height function $\mathbb{Z}R \to \mathbb{Z}$ allow us to define the graded dual module $N_w^* = \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}((N_w)_{-m}, \mathbb{C})$. Then as a left U-module S_w is defined by

$$S_w = U \otimes_{N_w} N_w^*.$$

This definition uses the left N_w-module structure on N_w^* given by $(xf)(n) = f(nx)$, $n, x \in N_w$, $f \in N_w^*$. The corresponding right N_w-module structure on N_w^* makes also S_w into a right N_w-module. It is an important fact that this extends to a right U-module structure on S_w. For an explicit proof of this, see [**So**].

The twisting functor T_w on the category of U-modules is then defined by

$$T_w(M) = \phi_w(S_w \otimes_U M),$$

when M is a U-module. Here ϕ_w is conjugation by an element in $\mathrm{Aut}(\mathfrak{g})$ corresponding to w.

The twisted Verma module $M^w(\lambda)$ is finally defined as

$$M^w(\lambda) = T_w(M(w^{-1}\cdot\lambda)).$$

We use here the dot action of W on $\mathfrak{h}^*$ given by $w\cdot\lambda = w(\lambda+\rho)-\rho$, ρ being half the sum of the positive roots.

2.3. Note that if $e \in W$ is the neutral element then T_e is the identity functor. In fact, $\mathfrak{n}_e = 0$ and so $S_e = U \otimes_k k = U$. Hence we have

(1) $$M^e(\lambda) = M(\lambda) \text{ for all } \lambda \in \mathfrak{h}^*.$$

Let $\mathcal{O}$ denote the BGG-category for $(\mathfrak{g}, \mathfrak{b})$. If $M \in \mathcal{O}$ we write $\mathrm{ch}M$ for the character of M. We let $D : \mathcal{O} \to \mathcal{O}$ denote the duality functor which satisfies $\mathrm{ch}DM = \mathrm{ch}M$ for all $M \in \mathcal{O}$. Then

(2) $$\mathrm{ch}\ M^w(\lambda) = \mathrm{ch}\ M(\lambda) \text{ for all } \lambda \in \mathfrak{h}^*,\ w \in W.$$

and

(3) $$DM^w(\lambda) \simeq M^{ww_0}(\lambda) \text{ for all } \lambda \in h^*,\ w \in W.$$

Here w_0 is the longest element in W.

For $\lambda \in \mathfrak{h}^*$ we set $W(\lambda) = \{w \in W \mid w(\lambda) \in \lambda + \mathbb{Z}R\}$. This is the Weyl group corresponding to the root system $R(\lambda) = \{\alpha \in R \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}\}$. We denote by $\mathcal{O}_\lambda$ the block in $\mathcal{O}$ consisting of all modules whose composition factors have highest weights in $W(\lambda)\cdot\lambda$. Then $\mathcal{O}$ decomposes into a direct sum of these blocks. When $\lambda, \mu \in \mathfrak{h}^*$ lie in the closure of the same Weyl chamber and $\lambda - \mu$ is integral then we have a translation functor $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$. We shall in particular make use of these in the case where λ belongs to the interior (i.e., λ is regular) and μ to exactly one wall of a Weyl chamber. If $s \in W(\lambda)$ is the reflection corresponding to this wall we denote by θ_s the "wall-crossing functor" $T_\mu^\lambda \circ T_\lambda^\mu$.

Assume that $\lambda \in \mathfrak{h}^*$ is a regular weight. Then the twisted Verma modules have the following properties with respect to translation:

If $\mu \in \lambda + \mathbb{Z}R$ belongs to the closure of the $W(\lambda)$-chamber containing λ then

(4) $$T_\lambda^\mu M^w(\lambda) \simeq M^w(\mu) \text{ for all } w \in W(\lambda).$$

Let $w \in W(\lambda)$ and let s be a simple reflection in $W(\lambda)$ such that $ws > w$. If $w^{-1} \cdot \lambda < sw^{-1} \cdot \lambda$ then we have an isomorphism

(5) $$M^w(\lambda) \simeq M^{ws}(\lambda).$$

Let $w, r \in W(\lambda)$ with r being a reflection in a wall of the $W(\lambda)$-chamber containing λ. If $w^{-1} \cdot \lambda > w^{-1}r \cdot \lambda$. Then we have a short exact sequence

(6) $$0 \to M^w(\lambda) \to \theta_r M^w(\lambda) \to M^w(r \cdot \lambda) \to 0.$$

It is proved in [**AL**] that the properties (1)-(6) characterize twisted Verma modules. In fact, the conditions in Theorem 5.1 in [**AL**] are much weaker (for instance (3) follows from the other conditions, see Corollary 5.1 in [**AL**]. Indeed, this was the only way we could prove (3)).

3. The quantum case

Set $k = \mathbb{Q}(q)$ with q an indeterminate and let $U_q = U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. As usual (see e.g. [**Ja**]) we denote the generators of this k-algebra by E_i, F_i, K_i and K_i^{-1}, $i = 1, \cdots, n$, and we let U_q^+, U_q^- and U_q^0 denote the subalgebras generated by the E_i's, the F_i's, and the $K_i^{\pm 1}$'s, respectively. Then $U_q = U_q^- U_q^0 U_q^+$. We set $B_q = U_q^0 U_q^+$.

Let $\lambda \in (k^\times)^n$. The Verma module for U_q with highest weight λ is defined just as in the classical case

$$M_q(\lambda) = U_q \otimes_{B_q} k_\lambda.$$

Here k_λ is the 1-dimensional B_q-module on which K_i acts as multiplication by λ_i and E_i acts as 0.

3.1. We shall now see how we can imitate the construction of twisted Verma modules mentioned in Section 2.

On U_q we have a natural $\mathbb{Z}R$- (respectively $\mathbb{Z}$-) grading in which E_i has degree α_i (respectively 1), F_i has degree $-\alpha_i$ (respectively -1) and $K_i^{\pm 1}$ has degree 0. For $\lambda \in \mathbb{Z}R$ (respectively $m \in \mathbb{Z}$) we denote the subspace in U_q consisting of elements of degree λ (respectively m) by $(U_q)_\lambda$ (respectively $(U_q)_m$). If N is a $\mathbb{Z}$-graded subalgebra of U_q we define N^* to be the graded dual of N, i.e.

$$N^* = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_k(N_{-m}, k).$$

This is a left and right N-module via $(nf)(x) = f(xn)$, respectively $(fn)(x) = f(nx)$, $n, x \in N, f \in N^*$.

For each $i = 1, \cdots, n$ we have a braid group operator R_i on U_q (we prefer the letter R instead of the more commonly used T for these operators because our twisting functors are denoted by T). We have $R_i((U_q)_\lambda) = (U_q)_{s_i\lambda}$ for all $\lambda \in \mathbb{Z}R$.

Let $w \in W$ and pick a reduced expression $w = s_{i_r} \cdots s_{i_1}$ for w. Set $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Then $\{\beta_1, \cdots, \beta_r\} = \{\beta \in R^+ \mid w(\beta) < 0\}$. We set

$$F_{\beta_j} = R_{i_1} \cdots R_{i_{j-1}}(F_{i_j}) \in (U_q)_{-\beta_j}$$

and define $U_q^-(w)$ to be the subspace of U_q^- spanned by the monomials

(1) $$F_{\beta_r}^{a_r} \cdots F_{\beta_2}^{a_2} F_{\beta_1}^{a_1}, \ a_j \in \mathbb{N}.$$

It is a fact [Ja, 8.24] that $U_q^-(w)$ is a subalgebra of U_q^- with basis consisting of the set of monomials in (1). This subalgebra depends only on w (not on the reduced expression we have picked). Moreover, if we set $w' = s_{i_r} w$ then (see e.g. [**DP**], 9.3)

$$q^{-(\beta_j,\beta_r)} F_{\beta_r} F_{\beta_j} - F_{\beta_j} F_{\beta_r} \in U_q^-(w') \tag{2}$$

for all $j = 1, \cdots, r-1$. Here $(\quad , \quad)$ is the usual W-invariant symmetric bilinear pairing on $\mathbb{Z}R$.

Note that (2) allows us to define an action of F_{β_r} (and hence a $k[F_{\beta_r}]$-module structure) on $U_q^-(w')$ by

$$ad_q(F_{\beta_r})(n) = [F_{\beta_r}, n]_q := q^{(\lambda,\beta_r)} F_{\beta_r} n - n F_{\beta_r} \tag{3}$$

for $n \in U_q^-(w')_\lambda$.

DEFINITION 3.1. The quantized semiregular module S_q^w associated with w is the left U_q^- and right $U_q^-(w)$-module

$$S_q^w = U_q \otimes_{U_q^-(w)} U_q^-(w)^*.$$

Note that for $w = e$ we have $U_q^-(e) = k$ and $S_q^e = U_q$.

3.2. We shall now consider the case $w = s_i$, the i'th simple reflection. Here $U_q^-(s_i) = k[F_i]$. We shall need the following

LEMMA 3.2. *The set $S = \{F_i^m \mid m \in \mathbb{N}\}$ is an Ore subset of U_q.*

PROOF. We must check that for every $u \in U_q$ and every $a \in \mathbb{N}$ we have

$$Su \cap U_q F_i^a \neq \emptyset \neq F_i^a U_q \cap uS.$$

This follows easily from the following relations

$$ad_q(F_i)^{1-a_{ij}}(F_j) = 0 \text{ for } i \neq j \tag{1}$$

(with a_{ij} being the (i,j)'th entry in the Cartan matrix for R and $ad_q(F_i)$ defined in 3.1(3) above),

$$[F_i, E_j] = 0 \text{ for } i \neq j, \tag{2}$$

and

$$E_i^{(r)} F_i^{(s)} = \sum_{j \geq 0} F_i^{(s-j)} \begin{bmatrix} K_i; 2j - r - s \\ j \end{bmatrix} E_i^{(r-j)} \tag{3}$$

(with notation as in [**Lu**]).

Note that (1) and (2) are among the defining relations for U_q and (3) (sometimes called Kac's formula) is a "higher version" of the defining relation $[E_i, F_i] = \begin{bmatrix} K_i;0 \\ 1 \end{bmatrix}$. □

Let now $U_{q(F_i)}$ denote the Ore localization of U_q at the set $\{1, F_i, F_i^2, \cdots\}$. This is clearly both a left and a right U_q-module. So is the quotient $U_{q(F_i)}/U_q$ and we have

PROPOSITION 3.3. *There exists an isomorphism of left U_q-modules and right $k[F_i]$-modules $S_q^{s_i} \simeq U_{q(F_i)}/U_q$.*

PROOF. Let us drop the index i from the notation. Write $f_m \in k[F]^*$ for the map given by $F^j \mapsto \delta_{j,m}$ for $j \in \mathbb{N}$. Then we define a k-linear map $U_{q(F)} \to S_q^s$ by

$$u/F^m \mapsto uF \otimes f_m,\ u \in U_q,\ m \in \mathbb{N}.$$

Since $Ff_m = f_{m-1}$ for $m > 0$ and $Ff_0 = 0$ this is a well defined map with kernel equal to U_q. It clearly induces the desired isomorphism. □

3.3. Returning to the case of a general $w \in W$ we resume the notation above. Lemma 3.2 implies that the sets $\left\{1, F_{\beta_j}, F_{\beta_j}^2, \cdots\right\}$, $j = 1, \cdots, r$ are Ore subsets of U_q. We let $U_{q(F_{\beta_j})}$ denote the corresponding Ore localization of U_q and set

$$S_q(F_{\beta_j}) = U_{q(F_{\beta_j})}/U_q.$$

We shall now prove a quantum analogue of Lemma 3.2.6 in [**Ar**]:

PROPOSITION 3.4. *Let $w \in W$ have reduced expression $w = s_{i_r} \cdots s_{i_1}$ and define $\beta_1, \beta_2, \cdots .\beta_r \in R^+$ as in 3.1. Then there exists an isomorphism of left U_q-modules and right $U_q^-(w)$-modules*

$$S_q^w \simeq S_q(F_{\beta_1}) \otimes_{U_q} \cdots \otimes_{U_q} S_q(F_{\beta_r}).$$

PROOF. We proceed by induction on r. For $r = 1$ the statement is Proposition 3.3. So assume $r > 1$ and set $w' = s_{i_r} w$. Writing F short for F_{β_r} we define a (right) $k[F]$-module structure on $U_q^-(w) \otimes_{U_q^-(w')} U_q^-(w')^*$ by

$$(n \otimes f)F = q^{(\lambda,\beta_r)}(nF \otimes f + n \otimes [F,f]_q) \tag{1}$$

for $n \in U_q^-(w)$, $f \in \left(U_q^-(w')^*\right)_\lambda$. Here $[F,f]_q$ is the linear map on $U_q^-(w')$ given by $x \mapsto f\left([F,x]_q\right)$. Note that (by (2 in 3.1) above) we have $[F,x]_q \in U_q^-(w')$ for all $x \in U_q^-(w')$.

To check that (1) is a well defined action we need to verify

$$(nx \otimes f)F = (n \otimes xf)F \text{ for all } x \in U_q^-(w')_\mu. \tag{2}$$

We first compute the left hand side (using 3.1 (3))

$$\begin{aligned}&(nx \otimes f)F = q^{(\lambda,\beta_r)}(nxF \otimes f + nx \otimes [F,f]_q) = \\ &q^{(\lambda,\beta_r)}(n(q^{(\mu,\beta_r)}Fx - [F,x]_q) \otimes f + nx \otimes [F,f]_q) = \\ &q^{(\lambda+\mu,\beta_r)}nF \otimes xf - q^{(\lambda,\beta_r)}n \otimes [F,x]_q f + q^{(\lambda,\beta_r)}n \otimes x[F,f]_q.\end{aligned}$$

On the other hand the right hand side of (2) equals (noting that $xf \in \left(U_q^-(w')^*\right)_{\lambda+\mu}$)

$$q^{(\lambda+\mu,\beta_r)}\left(nF \otimes xf + n \otimes [F,xf]_q\right).$$

Hence (2) comes from the following equality valid for all $y \in U_q^-(w')_\lambda$

$$-y[F,x]_q + [F,yx]_q = q^{(\mu,\beta_r)}[F,y]_q x.$$

We claim now that (1) leads to an isomorphism of left $U_q^-(w)$-modules

$$U_q^-(w)^* \simeq (U_q^-(w) \otimes_{U_q^-(w')} U_q^-(w')^*) \otimes_{k[F]} k[F]^*. \tag{3}$$

To prove this claim we first observe that (because of 3.1(1)) we may write elements of $U_q^-(w)^*$ as linear combinations of $f \cdot g$ with $f \in U_q^-(w')^*$, $g \in k[F]^*$. Here $f \cdot g$ is the linear map on $U_q^-(w)$ given by

$$(f \cdot g)(F^a n) = f(n)g(F^a),\ a \in \mathbb{N},\ n \in U_q^-(w').$$

Therefore the action of $U_q^-(w)$ on $f \cdot g$ is determined by the following two formulas

(4) $$x(f \cdot g) = (xf) \cdot g, \ x \in U_q^-(w'),$$

and

(5) $$F(f \cdot g) = q^{-(\lambda, \beta_r)}(f \cdot Fg) - [F, f]_q \cdot g, \ f \in (U_q^-(w')^*)_\lambda.$$

In fact, (4) is obvious and (5) follows from the computations (where $a \in \mathbb{N}$, $n \in U_q^-(w')_\mu$)

$$\begin{aligned} F(f \cdot g)(F^a n) &= f \cdot g(F^a n F) = f \cdot g(F^a q^{(\mu, \beta_r)} F n - F^a [F, n]_q) \\ &= q^{(\mu, \beta_r)} f(n) g(F^{a+1}) - f([F, n]_q) g(F^a). \end{aligned}$$

Define now $\phi : U_q^-(w)^* \to (U_q^-(w) \otimes_{U_q^-(w')} U_q^-(w')^*) \otimes_{k[F]} k[F]^*$ by

$$\phi(f \cdot g) = 1 \otimes f \otimes g.$$

Then (1), (4) and (5) imply that ϕ is a $U_q^-(w)$-homomorphism. It is therefore also an isomorphism and we have proved (3).

Using (3) we finally get

$$\begin{aligned} S_q^w = U_q \otimes_{U_q^-(w)} U_q^-(w)^* &\simeq U_q \otimes_{U_q^-(w)} (U_q^-(w) \otimes_{U_q^-(w')} U_q^-(w')^*) \otimes_{k[F]} k[F]^* \\ &\simeq U_q \otimes_{U_q^-(w')} U_q^-(w')^* \otimes_{U_q} U_q \otimes_{k[F]} k[F]^* \simeq S_q^{w'} \otimes_{U_q} S_q^{s_{i_r}}. \end{aligned}$$

The induction hypothesis now finishes the proof. □

3.4. The previous proposition shows that S_q^w has a right U_q-module structure. A priori this stucture might depend on the reduced expression for w. However, we now prove

PROPOSITION 3.5. *There exists an isomorphism of right U_q-modules*

$$S_q^w \simeq U_q^-(w)^* \otimes_{U_q^-(w)} U_q$$

with S_q^w equipped with the right U_q-module structure provided by Proposition 3.4.

PROOF. This is proved using arguments similar to the ones above so we shall just sketch the line of arguments. Set $w'' = ws_{i_1}$ and write F' short for F_{i_1}. With $N_q(w'') = R_{i_1}(U_q^-(w''))$ we then have (see [**DP**])

(1) $$[F', y]_q = F'y - q^{(\lambda, \beta_{i_1})} y F' \in N_q(w'')$$

for all $y \in N_q(w'')_\lambda$. Hence the following formula gives $N_q(w'')^* \otimes_{N_q(w'')} U_q^-(w)$ a left $k[F']$-module structure

(2) $$F'(f \otimes n) = q^{(\lambda, \beta_{i_1})}(f \otimes F'n - [F', f]_q \otimes n),$$

$f \in (N_q(w'')^*)_\lambda$ and $n \in U_q^-(w)$. This in turn leads to an isomorphism of right $U_q^-(w)$-modules

(3) $$U_q^-(w)^* \simeq k[F']^* \otimes_{k[F']} (N_q(w'')^* \otimes_{N_q(w'')} U_q^-(w)),$$

and hence to an isomorphism of right U_q-modules

(4) $$U_q^-(w)^* \otimes_{U_q^-(w)} U_q \simeq k[F']^* \otimes_{k[F']} (N_q(w'')^* \otimes_{N_q(w'')} U_q).$$

By induction on r (using Proposition 3.2 for the start) we see that the right hand side of (4) may be identified with $S_q(F') \otimes_{U_q} R_{i_1}(S_q^{w''}) \simeq S_q(F_{\beta_1}) \otimes_{U_q} S_q(F_{\beta_2}) \otimes_{U_q} \cdots \otimes_{U_q} S_q(F_{\beta_r})$. Conclusion by Proposition 3.4. □

3.5. Having verified that S_q^w has a natural U_q-bimodule structure we are ready to define the twisting functor associated to w on the category of (left) U_q-modules. Let us denote by R_w the automorphism of U_q given by $R_w = R_{i_r} \cdots R_{i_1}$ and if V is a U_q-module we write $R_w(V)$ for the U_q-module whose underlying k-space is V but with U_q-action given by $u \cdot v = R_w(u)v$, $u \in U_q$, $v \in V$.

DEFINITION 3.6. Let M be a U_q-module. Then the twisting functor T_w is defined by

$$T_w M = R_w(S_q^w \otimes_{U_q} M).$$

The twisted Verma module corresponding to $\lambda \in (k^\times)^n$ is

$$M_q^w(\lambda) = T_w M_q(w^{-1} \cdot \lambda).$$

Note that T_e is the identity functor. Hence we have immediately

(1) $$M_q^e(\lambda) = M_q(\lambda) \text{ for all } \lambda \in (k^\times)^n.$$

If we extend the reduced expression $w = s_{i_r} \cdots s_{i_1}$ to a reduced expression for w_0, $w_0 = s_{i_N} \cdots s_{i_{r+1}} s_{i_r} \cdots s_{i_1}$ and likewise extend the definition of β_j to the full range $j = 1, \cdots, N$ then it follows from (1) in Section 3.1 (with the order reversed, see [Ja]) that $U_q^-(w_0) = U_q^-$ is free as a left $U_q^-(w)$-module with basis $\left\{F_{\beta_{r+1}}^{a_{r+1}} \cdots F_{\beta_N}^{a_N} \mid a_j \in \mathbb{N}\right\}$. Denote by U_q^w the k-span of this basis. Then we have isomorphisms of U_q^0-modules (using Proposition 3.5)

$$S_q^w \otimes_{U_q} M_q(\\ lambda) \simeq U_q^-(w)^* \otimes_{U_q^-(w)} U_q \otimes_{U_q} U_q \otimes_{B_q} k_\lambda \simeq$$
$$U_q^-(w)^* \otimes_k U_q^w \otimes_k B_q \otimes_{B_q} k_\lambda \simeq U_q^-(w)^* \otimes_k U_q^w \otimes_k k_\lambda.$$

This allows us to determine the character of $M_q^w(\lambda)$. We get

(2) $$\text{ch} M_q^w(\lambda) = \text{ch} M_q(\lambda).$$

It follows from (2) that T_w preserves the BGG-category $\mathcal{O}_q$. Now we can proceed just as in the classical case from Section 1 to verify the properties analogous to (3)-(6). (For the analogue of (3) we should point out that we use the duality functor D on $\mathcal{O}_q$ given by $DM = \oplus(M_\mu)^*$ with U_q-action $(u \cdot f)(m) = f(S(\omega(u)m)$, $u \in U_q$, $m \in M$, $f \in DM$. Here ω is the automorphism of U_q from [Ja, 4.6] and S is the antipode on U_q).

REMARK 3.7. In the classical case we proved in [**AL**] that the properties (1) - (6) in Section 1 characterize twisted Verma modules (more precisely we proved that the weaker conditions in 5.1 of [**AL**] do). The same is true in our quantum case. One of the subtle points in the proof of this is to see that the endomorphism ring of $M_q^w(\lambda)$ is just k. The easiest way to check the quantum version of this is via specialization to the classical case, see Proposition 4.3 below.

4. Deformations and filtrations

We saw in [**AL**] that the construction of Verma modules by means of the twisting functors T_w is well suited for extensions of the ground ring $\mathbb{C}$. In particular, we used this to define Jantzen type filtrations of twisted Verma modules and we derived the corresponding sum formulas. In this section we shall see that a similar procedure works in the quantum case. Most of the proofs are completely analogous to the ones presented in [**AL**] and we omit the details.

4.1. Let X be an indeterminate and set $\tilde{k} = k(X)$. The quantum group $U_q(\tilde{k})$ over $\tilde{k}$ is defined exactly as U_q; i.e. using the same generators and relations but replacing k by $\tilde{k}$. Alternatively, $U_q(\tilde{k}) = U_q \otimes_k \tilde{k}$. We may also for each $\lambda \in (\tilde{k}^\times)^n$ define a Verma module $M_q(\lambda)_{\tilde{k}}$ and twisted Verma modules $M_q^w(\lambda)_{\tilde{k}}$, $w \in W$ with highest weight λ just like we did for U_q in Section 2.

Consider the local ring $A = k[X]_{(X-1)} \subset \tilde{k}$. and let $A^\times$ denote the units in A. We have then the quantum group $U_q(A)$ over A with Verma modules $M_q(\lambda)_A$ and twisted Verma modules $M_q^w(\lambda)_A$ for all $\lambda \in (A^\times)^n$ and $w \in W$. These modules are A-forms of the corresponding modules for $U_q(\tilde{k})$. Moreover, when we consider k as an A-algebra via the specialization $X \mapsto 1$ then we have

$$M_q^w(\lambda)_A \otimes_A k \simeq M_q^w(\bar{\lambda})$$

where $\bar{\lambda} \in (k^\times)^n$ is the specialization of λ.

4.2. We shall first look at the case where the underlying Lie algebra is $\mathfrak{sl}_2$. In this case we denote the generators (for U_q as well as $U_q(\tilde{k})$) by E, F, K and K^{-1}. For each $\lambda \in \tilde{k}^\times$ we have just two twisted Verma modules, namely

$$M_q^e(\lambda)_{\tilde{k}} \simeq M_q(\lambda)_{\tilde{k}} \text{ and } M_q^s(\lambda)_{\tilde{k}} \simeq DM_q(\lambda)_{\tilde{k}},$$

where s denotes the non-trivial element in W.

The universal property of Verma modules gives us a natural homomorphism

$$\phi_\lambda : M_q(\lambda)_{\tilde{k}} \to DM_q(\lambda)_{\tilde{k}}.$$

Let $v_0 \in M_q(\lambda)_{\tilde{k}}$ be a highest weight vector and set $v_i = F^{(i)}v_0$ (using the standard divided power notation $F^{(i)} = \frac{F^i}{[i]!}$ as in [**Ja**]). Then $\{v_i\}$ is the usual basis for $M_q(\lambda)_{\tilde{k}}$. An easy computation shows that in terms of the corresponding dual basis $\{v_i^*\}$ in $DM_q(\lambda)_{\tilde{k}}$ we have

(1) $$\phi_\lambda(v_i) = a_i v_i^*, \ i \in \mathbb{N},$$

where $a_i = (-1)^i q^{i(i-1)} \lambda^{-i} \left[{\lambda \atop i}\right]$, $\left[{\lambda \atop i}\right] = \prod_{j=1}^{i} \frac{\lambda q^{1-j} - \lambda^{-1} q^{j-1}}{q^j - q^{-j}}$. Note that if $\lambda = q^r$ for some $r \in \mathbb{Z}$ then $\left[{\lambda \atop i}\right] = \left[{r \atop i}\right]$ in the notation from [**Ja**].

It follows from (1) that ϕ_λ is an isomorphism if and only if $\lambda \notin \{\pm q^r \mid r \in \mathbb{N}\}$. On the other hand, if $\lambda = \pm q^r$ for some $r \in \mathbb{N}$ then we see that $\mathrm{Ker}(\phi_\lambda) \simeq M_q(\pm q^{-r-2})_{\tilde{k}}$ and $\mathrm{Coker}(\phi_\lambda) \simeq DM_q(\pm q^{-r-2})_{\tilde{k}} \simeq M_q(\pm q^{-r-2})_{\tilde{k}}$. In other words, we have a 4-term exact sequence

$$0 \to M_q(\pm q^{-r-2})_{\tilde{k}} \to M_q(\pm q^r)_{\tilde{k}} \to M_q^s(\pm q^r)_{\tilde{k}} \to M_q(\pm q^{-r-2})_{\tilde{k}} \to 0$$

Suppose $\lambda \in A^\times$. Then the formula (1) shows that the homomorphism $\phi_\lambda : M_q(\lambda)_A \to M_q^s(\lambda)_A$ is an isomorphism if and only if $a_i \in A^\times$ for all i, i.e. if and only if $\left[{\lambda \atop i}\right]$ is not divisible by $X - 1$ for all i.

Consider the special case where $\lambda = \pm q^r X$ for some $r \in \mathbb{Z}$. If $r < 0$ then the above remarks show that ϕ_λ is an isomorphism. However, if $r \geq 0$ then $X - 1$ divides $\left[{\lambda \atop i}\right]$ for all $i > r$. This leads to the exact sequence

$$0 \to M_q(\pm q^r X)_A \to M_q^s(\pm q^r X)_A \to M_q(\pm q^{-r-2}) \to 0.$$

Here the first map is $\phi_{\pm q^r X}$ whose cokernel $M_q^s(\pm q^{-r-2}X)_A/(X-1)M_q(\pm q^{-r-2}X)_A$ we have identified with $M_q(\pm q^{-r-2})$.

Analogous arguments lead to a homomorphism $\psi_\lambda : DM_q(\lambda)_{\tilde{k}} \to M_q(\lambda)_{\tilde{k}}$ with similar properties.

4.3. Let us now return to the case of a general $\mathfrak{g}$. Here we will consider a fixed weight $\lambda \in \mathbb{Z}^n$ corresponding to the character $U_q^0 \to k$ which takes K_i into $q^{d_i\lambda_i}$. We thus restrict ourselves to integral weights and we have chosen all signs to be $+1$. It is standard how to generalize from this to the characters $K_i \to \pm q^{d_i\lambda_i}$. Then we shall study the character $q^\lambda X$ of U_A^0 (and of $U_{q\tilde{k}}^0$) given by $K_i \mapsto q^{d_i\lambda_i}X$.

Fix also $w \in W$ and a reduced expression $w = s_{i_r}\cdots s_{i_1}$. Extend to a reduced expression $s_{i_N}\cdots s_{i_{r+1}}s_{i_r}\cdots s_{i_1}$ for w_0 and set (as in [**AL**] but not quite as in Section 3.1)

$$\beta_j = \begin{cases} -ws_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j \le r \\ ws_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}), & \text{if } j > r. \end{cases}$$

Then $\{\beta_1,\cdots,\beta_N\} = R^+$ and $\{\beta_1,\cdots,\beta_r\} = \{\beta \in R^+ \mid w^{-1}(\beta) < 0\} = R^+(w)$.

For each $i = 1,\cdots,N$ we have a minimal parabolic subalgebra $U_q(i)$ of U_q, namely the one generated by B_q together with F_i. Then the corresponding Verma module for $U_q(i)$ is $M_{q,i}(\lambda) = U_q(i)\otimes_{B_q}k_\lambda$ and the twisted Verma module $M_{q,i}^{s_i}(\lambda) = R_i(U_q(i)\otimes_{k[F_i]}k[F_i]^*\otimes_{U_q(i)}M_{q,i}(s_i\cdot\lambda))$. There are also analogues of these modules over A and $\tilde{k}$ relative to the character $q^\lambda X$.

The $\mathfrak{sl}_2$-theory in Section 4.2 leads to natural $U_q(i)$-homomorphisms

$$\phi_{\lambda,i} : M_{q,i}(q^\lambda X)_A \to M_{q,i}^{s_i}(q^\lambda X)_A$$

and

$$\psi_{\lambda,i} : M_{q,i}^{s_i}(q^\lambda X)_A \to M_{q,i}(q^\lambda X)_A$$

with properties similar to the homomorphisms ϕ_λ and ψ_λ in 4.2.

Exploring the fact that $T_w = T_{ws_{i_1}} \circ T_{s_{i_1}}$ (see Proposition 3.4) together with the relations between twisted Verma modules for $U_q(i_1)$ and for U_q (see Section 6.6 in [**AL**]) we obtain (as in loc. cit. Section 7) a sequence of natural $U_q(A)$-homomorphisms

$$M_q^w(q^\lambda X)_A \to M_q^{ws_{i_1}}(q^\lambda X)_A \to \cdots \to M_q^{ww_0}(q^\lambda X)_A$$

Each homomorphism in this sequence is induced by some $\phi_{\lambda,i}$ or $\psi_{\lambda,i}$. We denote the composite by $\phi^w(\lambda)$ and define

$$M_q^w(q^\lambda X)_A^j = \left\{m \in M_q^w(q^\lambda X)_A \mid \phi^w(\lambda)(m) \in (X-1)^j M_q^{ww_0}(q^\lambda X)_A\right\}.$$

Taking the image of this filtration under the specialization map $M_q^w(q^\lambda X)_A \to M_q^w(q^\lambda)$ (induced by $X \mapsto 1$) produces a filtration $(M_q^w(q^\lambda)^j)_{j\ge 0}$ of $M_q^w(q^\lambda)$.

In analogy with 7.1 in [**AL**] we get

PROPOSITION 4.1. *Let λ, w be as above. Then $M_q^w(q^\lambda)$ has a Jantzen filtration*

$$M_q^w(q^\lambda) = M_q^w(q^\lambda)^0 \supset M_q^w(q^\lambda)^1 \supset \cdots \supset 0$$

such that $M_q^w(q^\lambda)/M_q^w(q^\lambda)^1$ is isomorphic to the image of the composite $M_q^w(q^\lambda) \to M_q^{ws_{i_1}}(q^\lambda) \to \cdots \to M_q^{ww_0}(q^\lambda)$ and

$$\sum_{j\ge 1}\mathrm{ch}M_q^w(q^\lambda)^j = \sum_{\beta\in R^+(w)}(\mathrm{ch}M_q(q^\lambda) - \mathrm{ch}M_q(q^{s_\beta\cdot\lambda})) + \sum_{\beta\in R^+\setminus R^+(w)}\mathrm{ch}M_q(q^{s_\beta\cdot\lambda}).$$

REMARK 4.2. i) This proposition generalizes the Jantzen sum formula for ordinary quantized Verma modules (cf. 4.4.17 in [**Jo**]).

ii) Note that (just as in the classical case) we sometimes have $M_q^w(q^\lambda)^1 = M_q^w(q^\lambda)$. This is connected with the fact that twisted Verma modules do not in general have simple socles and heads. An illustration of this is the B_2-case treated in 7.4 of [**AL**], where we used the sum formula to compute all filtration layers (the very same computations apply in the quantum case). Further examples and results on the structure of twisted Verma modules can be found in [**St**] (in the classical case). Their socles and heads are not known in general.

4.4. Finally, let us point out that quantized twisted Verma modules are deformations of their classical counter parts. Set namely, $A' = \mathbb{C}[q]_{(q-1)} \subset k$ and let $U_{A'}$ be the A'-subalgebra of U_q generated by $E_i^{(r)}$, $F_i^{(r)}$, K_i and K_i^{-1}, $i = 1, \cdots, n$, $r \in \mathbb{N}$. Then we have (for $\lambda \in \mathbb{Z}^n$ as before) A'-forms $M_{A'}^w(\lambda)$ of the twisted Verma modules $M_q^w(\lambda)$. To construct these we proceed as in Section 3 except that we work consistently with divided powers instead of ordinary powers. For instance $U_{A'}^w$ is the A'-subalgebra of $U_{A'}$ spanned by the set $\left\{F_{\beta_r}^{(a_r)} \cdots F_{\beta_1}^{(a_1)} \mid a_j \in \mathbb{N}\right\}$, cf. (1) in Section 3.1.

The twisted Verma modules for $U_{A'}$ are free over A' and satisfy

$$M_{A'}^w(\lambda) \otimes_{A'} k \simeq M_q^w(q^\lambda) \text{ and } M_{A'}^w(\lambda) \otimes_{A'} \mathbb{C} \simeq M^w(\lambda).$$

Here $\mathbb{C}$ is considered as A'-algebra via the specialization $q \mapsto 1$.

As an application of this observation we record

PROPOSITION 4.3. *For all $\lambda \in \mathbb{Z}^n$ and $w \in W$ we have* $\mathrm{End}_{U_q}(M_q^w(q^\lambda)) \simeq k$.

PROOF. Clearly, $\mathrm{End}_{U_{A'}}(M_{A'}^w(\lambda)) \otimes_{A'} \mathbb{C} \subseteq \mathrm{End}_{U(\mathfrak{g})}(M^w(\lambda))$. However, the latter ring is just $\mathbb{C}$ by Corollary 6.3 in [**AL**]. Hence also

$$\mathrm{End}_{U_q}(M_q^w(q^\lambda)) \simeq \mathrm{End}_{U_{A'}}(M_{A'}^w(\lambda)) \otimes_{A'} k$$

is 1-dimensional. □

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Contemporary Mathematics
Volume **325**, 2003

On tameness of the Hecke algebras of type B

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Abstract. We conjecture that Hecke algebras have a block of tame representation type only if $q = -1$ and prove that a Hecke algebra of type B has tame representation type if and only if $q = -1$ and $n = 2$. As a related topic, we also develop theory of the Green correspondence for Hecke algebras of general type.

1. Introduction

Based on [**AM1**] and [**AM2**] we have determined when a Hecke algebra of classical type has finite representation type [**A4**]. Let F be an algebraically closed field of characteristic ℓ and assume that $q \in F^\times$ is a primitive e^{th} root of unity. Let W be a finite Weyl group of classical type, $\mathcal{H}_W(q)$ the associated Hecke algebra, which is defined over F. If $q = 1$ then $\mathcal{H}_W(q) = FW$ has tame representation type if and only if ℓ^2 does not divide $|W|$. Now assume that $e \geq 2$. Let $P_W(x) = \sum_{w \in W} x^{l(w)}$ be the Poincare polynomial of W. Then $\mathcal{H}_W(q)$ has finite representation type if and only if $(x-q)^2$ does not divide $P_W(x)$. This is a natural q-analogue of the old result of Higman [**H**] applied to Weyl groups. See [**A4**] for the details.

Let $F[X]$ be a polynomial ring. A finite dimensional F-algebra A is said to have tame representation type, if for each positive integer d there are finitely many $(A, F[X])$-bimodules $M_1, \ldots, M_{n_d}$ which are free as right $F[X]$-modules such that all but finite number of d-dimensional indecomposable A-modules M are of the form $M \simeq M_i \otimes_{F[X]} F[X]/(X - \lambda)$, for $1 \leq i \leq n_d$ and $\lambda \in F$. As is well-known, an Artinian algebra of infinite representation type has either tame representation type or wild representation type. This is a famous theorem of Drozd. See [**Dr**], [**C**] or [**E**, I.4.6].

As we have given a criterion for finite representation type, our next step is to determine when $\mathcal{H}_W(q)$ has tame representation type. In order to have an insight, let us recall the group algebra case again. In [**BD**], the authors proved that a group algebra has tame representation type if and only if the base field has characteristic 2 and the Sylow 2-subgroup is one of dihedral, semidihedral or generalized quaternion groups. On the other hand, the q-analogue philosophy suggests that if $q \neq 1$ then

1991 *Mathematics Subject Classification.* 20C08, 16G60.
Key words and phrases. Hecke algebra, tame representation type.