CONTEMPORARY MATHEMATICS

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Lie Algebras and Related Topics

Proceedings of a Research Conference
held May 22–June 1, 1988

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Georgia Benkart and
J. Marshall Osborn, Editors

AMERICAN MATHEMATICAL SOCIETY • PROVIDENCE, RHODE ISLAND

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PREFACE

During the academic year 1987-1988 the University of Wisconsin, Madison hosted a Special Year of Lie Algebras. A workshop in August 1987 inaugurated the year's activities, and a conference on Lie algebras and related topics in May 1988 marked its end. This volume contains the proceedings of the concluding conference, which featured lectures on Lie algebras of prime characteristic, algebraic groups, combinatorics and representation theory, and Kac-Moody and Virasoro algebras. Many of the facets of recent research on Lie theory are reflected in the papers presented here. The diversity that gives Lie theory its richness and relevance has also made it difficult for us to organize the papers by topic. For that reason we have chosen to arrange the papers alphabetically by the author's name.

In 1984 Richard Block and Robert Wilson announced the classification of the finite dimensional restricted simple Lie algebras over an algebraically closed field of characteristic $p > 7$. Their announcement provided the impetus for us to bring together researchers working on the long-standing problem of determining the finite dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 7$. That problem now appears to be much closer to a solution as a result of the work of the participants during the special year and afterwards, particularly that of Helmut Strade.

We would like to express our appreciation to the National Science Foundation for its support through grant #DMS-87-02928 which made the workshop, conference, and other events of the special year possible. Funds from the grant also enabled short-term visitors to come to Madison to give special seminars and colloquia and to participate in informal discussions. They joined a group of long-term visitors who spent most of the year in Madison. In addition, we thank our colleagues in the Mathematics Department of the University of Wisconsin for committing the department's resources to visitors for the special year. We are grateful too to the referees who helped in the preparation of the volume and to Diane Reppert for her excellent work in typing many of the manuscripts.

Georgia Benkart and J. Marshall Osborn

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ABSTRACTS OF TALKS

ISOTROPIC LIE ALGEBRAS OF TYPE D_4

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Suppose k is a field of characteristic 0. In recent work G.B. Seligman has shown how to construct all isotropic central simple Lie algebras of type D_4. In this talk, we describe a different construction of the isotropic D_4's. Given a 4-dimensional commutative associative separable algebra $B \setminus k$ and a scalar $\mu \neq 0 \in k$, we construct (using earlier work of J. Faulkner and the speaker) an 8-dimensional algebra with involution $(B \oplus vB, -)$ from B and μ, called a quartic Cayley algebra and then apply the Kantor Lie algebra construction to obtain an isotropic Lie algebra $K(B, \mu)$ of type D_4. We give a structural characterization of quartic Cayley algebras and use it to show that any isotropic D_4 is obtained from our construction or is isomorphic to the Lie algebra of a quadratic form. Next, using work of H.P. Allen, we give necessary and sufficient conditions for two algebras $K(B, \mu)$ and $K(B', \mu')$ to be isomorphic. We also show that our construction yields a class of D_4's that are not obtained from the classical constructions studied by N. Jacobson and by H.P. Allen.

THE ROGERS-RAMANUJAN IDENTITIES: BACKGROUND AND MOTIVATION

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Among Ramanujan's most famous discoveries are the Rogers-Ramanujan identities. Indeed Hardy says of them, "It would be difficult to find more beautiful formulae than the Rogers-Ramanujan identities". We discuss some of the history of these results as well as current research. We discuss in detail Rodney Baxter's independent discovery and proof of these results for application in statistical mechanics. By following Baxter's path we obtain reasonable motivation for a variation of the proof given by Rogers and Ramanujan.

SPECIAL FUNCTIONS IN AN ALGEBRAIC SETTING

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Macdonald's identities are multivariable extensions of Jacobi's triple product for the theta function. In one variable, Ramanujan and Bailey found extensions of the triple product identity. Just as the theta function is a discrete version of the normal integral, Ramanujan's sum is a discrete version of a beta integral, and Bailey's sum can be thought of as a discrete version of a much more complicated beta integral. One way of trying to extend Macdonald's identities is to add the extra freedom that exists when going from normal to beta integrals. S. Milne did this for A_n, and R. Gustafson has recently done this for $\widehat{B}_n$ and C_n. These identities extend Bailey's very well poised sum and are sufficiently rich to contain the Macdonald identities for all the infinite families, including BC_n and both versions of B_n and C_n. Gustafson has found the corresponding identity for G_2, but it does not contain the Macdonald identity for $\widehat{G}_2$, so further work needs to be done.

In a different direction, between the Macdonald identities for G_2 and $\widehat{G}_2$, where infinite products are on base q for G_2 and on base q for the short roots of $\widehat{G}_2$ but base q^3 for the long roots of $\widehat{G}_2$, there is the example with the products for the short roots on base q and the long roots on base q^2. The corresponding Laurent series has two nonvanishing orbits in the fundamental region. It seems likely this is general. For example, it is probably true that if one separates the roots somehow (say by lengths) and uses different bases for roots in different groups, the only cases where only one orbit under the Weyl group survives is when one has a classical affine root system. The work on $G_{2,2}$ is joint with Dennis Stanton, and is still in progress.

SOME REMARKS ON GENERALIZED INTERSECTION MATRIX ALGEBRAS AND A CONJECTURE OF P. SLODOWY

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If L is a G.I.M. algebra attached to the indecomposable matrix A, then P. Slodowy has shown that there is a homomorphism from L onto a subalgebra S of a Kac-Moody Lie algebra. Here S is the fixed point set of an involution of the Kac-Moody algebra. He showed this map is an isomorphism when the matrix A is orientable. We indicate that this is also the case when the matrix A is non-orientable, and in fact present a general result on generators and relations for fixed point sets of certain kinds of involutions of Kac-Moody algebras. This is a version of the Gabber-Kac Theorem for these algebras.

We also get that S has a filtration such that the associated graded algebra is isomorphic to the positive part of the Kac-Moody algebra, and using previous results of Benkart and Moody, get that these algebras are centrally closed. Finally, we present some examples showing some of these fixed point algebras - coming from Kac-Moody algebras of infinite type - have unique simple finite-dimensional factors.

A CHARACTERIZATION OF LIE ELEMENTS AT PRIME CHARACTERISTIC

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Recall that the free Lie algebra on a set X can be constructed as the Lie subalgebra $L(V)$ of $(TV)^-$ generated by V, where TV is the tensor algebra of a vector space V with basis X; the elements of $L(V)$ are called Lie elements. A classical theorem of Friedrichs, valid only at characteristic 0, states that $x \in TV$ is a Lie element if and only if x is primitive, i.e., $\Delta x = x \otimes 1 + 1 \otimes x$ where Δ is the comultiplication in the Hopf algebra TV. The graded dual $(TV)^*$ to TV (for simplicity assuming V is finite dimensional) is the shuffle algebra ShV^*, with multiplication given by shuffle product. Equivalent to Friedrichs' theorem is the dual version of Ree: at characteristic 0, x is a Lie element if and only if $x \in ((Ker\varepsilon^*) + (A^+)^2)^\perp$, i.e., x is annihilated by 1_A and the square of the augmentation ideal A^+ of A, where A denotes $(TV)^*$ ($= ShV^*$). This result is generalized to arbitrary characteristic using the divided power structure (coming from the divided powers $\gamma_n b = b^n/n!$ over Z) on the shuffle algebra. Let $\Gamma^2 A$ denote the divided power square of A^+, spanned by all products ab and divided powers $\gamma_n a$ ($n > 1$; $a, b \in A^+$). Call x ($\in TV$) Γ-primitive if $x \in ((Ker\varepsilon^*) + \Gamma^2 A)^\perp$.

Theorem. At arbitrary characteristic, x is a Lie element if and only if x is Γ-primitive.

STRUCTURE OF CERTAIN SIMPLE LIE ALGEBRAS OF CHARACTERISTIC 3

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Over a field k of characteristic 3 there exist simple Lie algebras that are not analogues of known simple Lie algebras of other characteristics. The structure of some of these algebras is discussed.

The algebras $L(\varepsilon)$, $\varepsilon \in k \setminus \{0\}$, of Kostrykin's parametric family of ten-dimensional simple Lie algebras, are defined as subalgebras of a contact algebra. $L(-1)$ is classical. Some distinctive properties of $L(1)$ are pointed out.

Using a construction due to Faulkner, Freudenthal triple systems unique to characteristic 3 are used to construct simple Lie algebras. Among these algebras is one of dimension 29. It contains a subalgebra S isomorphic to $L(1)$ and an S-module whose existence shows that $L(1)$ is isomorphic to a subalgebra of an algebra of type C_4.

Finally, an 18-dimensional simple Lie algebra discovered by M. Frank is discussed. It is shown that it contains a subalgebra isomorphic to $L(1)$, is a subalgebra of a contact algebra, and is the only restricted algebra in a certain class of simple algebras of dimension $2 \cdot 3^n$ for $n \geq 2$.

SOME PROBLEMS IN REPRESENTATION THEORY

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In this survey talk, representation theory was described as the search for realizations of mathematical objects such as Lie algebras, Lie groups, algebraic groups, finite groups etc. in terms of linear algebra. The basic problem of classification of finite dimensional irreducible representations of semisimple Lie algebras over the field of complex numbers was discussed. H. Weyl's results that these representations are determined by their highest weights, and his formula for their characters were examined, along with some remarks on how these ideas have been influential in more recent work. In particular, the classification using a version of highest weights of the irreducible representations of Chevalley groups and Lie algebras associated with them over algebraically closed fields of characteristic $p > 0$ was sketched, along with similar results on the classification of irreducible modular representations of finite Chevalley groups. In conclusion, Alperin's conjecture concerning a theory of weights which would classify irreducible modular representations of arbitrary finite groups, and Lusztig's conjecture on the irreducible Brauer characters of finite Chevalley groups, were stated.

THE GELFAND-GRAEV REPRESENTATIONS OF FINITE CHEVALLEY GROUPS - BESSEL FUNCTIONS OVER FINITE FIELDS

Charles Curtis

The Gelfand-Graev representation Γ of a finite Chevalley group G is an induced representation $\Gamma = ind_U^G \psi$, where ψ is a linear character in general position of the maximal unipotent subgroup U. The Hecke algebra H_Γ associated with Γ is known to be commutative. Its irreducible representations correspond to the irreducible components of Γ. In this talk, it was conjectured that each irreducible representation of H_Γ factors through the group algebra of some maximal torus of G. This was proved by Gelfand and Graev for the groups $SL_2(F_q)$, and by Chang for the groups $GL_3(F_q)$, with q odd in both cases. In the talk, a proof of the conjecture for the groups $SL_2(F_q)$ was sketched, using the Davenport-Hasse theorem on Gauss sums to establish a crucial identity for the case of nonsplit tori. As Gelfand and Graev had observed, the functions describing the representations of H_Γ in this case are analogous to the contour integral formulas for Bessel functions.

CALCULUS OF SOME FUNCTIONS ON THE ROOT LATTICE

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We discuss the interplay of various coordinatizations of the root lattice: the root, weight, capacity, and extraneous coordinates. Various combinatorial structures are assigned to each element of the root lattice as well: a weight space, i-blocks, blocks, and a zero block. A retraction, the bracketing function, from the positive cone of the root lattice onto its Kostant cone comes into play.

Using these coordinates and structures, in a completely elementary way, we obtain deep results about Kostant's partition function, both reductive in nature using the bracketing function, and inductive results using a duality result on i-blocks, as well as a zero block formula.

Applications to the representation theory of semisimple Lie algebras are noted.

KAZHDAN LUSZTIG POLYNOMIALS AND RELATED TOPICS

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This talk is divided into three parts.

I. *Survey of Kazhdan-Lusztig theory.* Here we discuss the impact of Kazhdan-Lusztig polynomials on some interesting problems in Lie theory. These include: (i) determination of multiplicities in Jordan-Hölder series of Verma modules, (ii) geometry of Schubert varieties in generalized flag manifolds, (iii) primitive ideal spectrum of enveloping algebras. Results prior to K-L papers enable us to put the K-L papers in proper perspective. In order to define the so-called Kazhdan-Lusztig polynomials (K-L polynomials), one has to consider the structure of Hecke algebras and a special involution on them; this is discussed in brief.

II. *Parabolic theme.* Here we briefly discuss the notion of a Hecke module corresponding to parabolic subsystems (of Coxeter groups, algebraic groups, Lie algebras, etc.). One then gets a relative version of K-L polynomials. These are very different in some cases and should be interesting in their own right. A general idea for this theme is to enable one to get information on original polynomials.

III. *A conjecture.* Here we look closely into the structure of Hecke algebras and formulate a conjecture for a closed formula for K-L polynomials. We also discuss the evidence in support of this conjecture.

BANACH STRUCTURES ON LOOP GROUPS

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The purpose of this talk is twofold. First we want to construct Banach manifolds X of Grassmannian type such that $AutX$ is a Banach Lie group of type A_∞, C_∞, or D_∞. The construction of X generalizes a finite dimensional procedure (see e.g. Loos' Irvine Lecture Notes) and works for Banach Jordan pairs for which the quasi-invertible elements are dense. The elements in the corresponding group $AutX$ have the usual "fourfold decomposition property" and $LieAutX$ is realized by polynomial maps of degree ≤ 2.

Inside X we consider $X^{(n)}$ which is defined as usual (see e.g. Segal-Wilson, IHES). Moreover, we define $(AutX)^{(n)}$ as the stabilizer of $X^{(n)}$. We show that $(AutX)^{(n)}$ is essentially isomorphic to the semidirect product of $s\ell(n, A)$, where A is a naturally defined Banach algebra of functions on the unit circle S^1, and $s\ell(2, \mathbb{R})$ where its action on $X^{(n)}$ is induced from its natural action on S^1 by diffeomorphisms.

Finally, in the second part we consider $s\ell(2, A)$, where $A \equiv$ Fourier transform of $L^1(\mathbb{R})$ and show how to derive (via a "generalized Riemann-Hilbert problem") the basic equation for all AKNS-systems. It is mentioned that by this procedure one obtains essentially all L^1-initial conditions. (Most of the work on which this talk is based is jointly with E. Neher and/or J. Szmigielski.)

DERIVATIONS AND ONE-DIMENSIONAL ABELIAN EXTENSIONS OF KAC-MOODY ALGEBRAS

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In this talk we introduce new methods and results pertaining to the cohomology theory of associative algebras with involution. Since universal enveloping algebras of contragredient algebras belong to this category, we obtain in particular general statements regarding cohomology groups of Kac-Moody algebras.

Let $\mathcal{G}$ be a not necessarily symmetrizable Kac-Moody algebra with Cartan subalgebra $\mathcal{H}$ and simple root vectors $e_1, \ldots, e_n, f_1, \ldots, f_n$. The involution of $U(\mathcal{G})$ which leaves $\mathcal{H}$ invariant and exchanges the e_i and f_i will be denoted by ω. Using the Shapovalov bilinear form in conjunction with the general results alluded to in the above, we obtain

Theorem. Let λ, $\mu \in \mathcal{H}^*$ and suppose that M is an object of the B-G-G category $\mathcal{O}$.

(1) $Ext^n_{U(\mathcal{G})}(M, L(\lambda)) \cong (Tor^{U(\mathcal{G})}_n(L(\lambda), M))^* \ \forall n \geq 0$, where $L(\lambda)$ has the structure of a right $U(\mathcal{G})$-module by setting $m \cdot u := \omega(u) \cdot m$.

(2) $Ext^n_{U(\mathcal{G})}(L(\lambda), L(\mu)) \cong Ext^n_{U(\mathcal{G})}(L(\mu), L(\lambda)) \ \forall n \geq 0$.

Let $\mathcal{G}'$ denote the derived algebra of $\mathcal{G}$ and suppose that $\mathcal{G}$ is associated to an $(n \times n)$-matrix of rank ℓ. For any homomorphism $\lambda : \mathcal{G} \longrightarrow F$ of Lie algebras let F_λ denote the corresponding one-dimensional $\mathcal{G}$-module. The following theorem generalizes several well-known classical results concerning derivations and central extensions of finite dimensional semisimple Lie algebras:

Theorem.

(1) $H^1(\mathcal{G}, \mathcal{G})$ has dimension $(n - \ell)^2$.

(2) $H^2(\mathcal{G}', F) = (0)$.

(3) $H^2(\mathcal{G}, F_\lambda) = \begin{cases} (0) & \text{for } \lambda \neq 0 \\ \Lambda^2(\mathcal{G} \setminus \mathcal{G}')^* & \text{otherwise.} \end{cases}$

THE EXCEPTIONAL AFFINE ALGEBRA $E_8^{(1)}$ AND TRIALITY

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A striking feature of the representation theory of affine algebras is the existence of a boson-fermion correspondence for some representations of some algebras. It means that one has two independent constructions, one using Heisenberg algebras and vertex operators, the other using Clifford algebras, and an isomorphism between the two pictures. For example, one has such a situation for the four representations of $D_\ell^{(1)}$ corresponding to the endpoints of the Dynkin diagram. In this work, which is joint with Igor B. Frenkel and John F. X. Reis, we have provided a boson-fermion correspondence for $E_8^{(1)}$ involving four versions of the basic representation, one homogeneously graded, three with "$k - p$" gradings. Their direct sum is naturally described in both pictures in terms of $D_4^{(1)}$ representations, and the principle of triality for $D_4^{(1)}$ plays an important role. We strongly contrast the aspects; vertex vs. spinor, finite dimensional vs. affine, and D_4 vs. E_8. The work has applications to conformal field theory.

THE CLASSIFICATION OF LIE ALGEBRA MODULES WITH FINITE DIMENSIONAL WEIGHT SPACES

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Let $\mathcal{G}$ be a finite dimensional, complex reductive Lie algebra, and let $\mathcal{U}(\mathcal{G})$ be its enveloping algebra. If $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$, then let $\mathcal{M}(\mathcal{G},\mathcal{H})$ denote the category of all finitely generated $\mathcal{U}(\mathcal{G})$-modules with finite dimensional $\mathcal{H}$-weight spaces. In this abstract we give a brief sketch of a classification of irreducible modules in $\mathcal{M}(\mathcal{G},\mathcal{H})$. Details will appear in [Fe1] and [Fe2]. In [Fe1] the problem of classifying irreducible modules in $\mathcal{M}(\mathcal{G},\mathcal{H})$ is reduced to the classification of "torsion free" irreducible modules of simple Lie algebras. A module $M \in \mathcal{M}(\mathcal{G},\mathcal{H})$ is said to be a torsion free module, if for every $x \in \mathcal{G}\setminus\mathcal{H}$ and $m \in M \setminus (0)$, $dim_{\mathbb{C}}\mathbb{C}[x]\cdot m = \infty$. We show that a simple Lie algebra admits a torsion free module if and only if the algebra is of type A or C. In what follows $\mathcal{G}$ will denote a simple Lie algebra of type A or C, and n will denote the rank of $\mathcal{G}$. Next we classify all primitive ideals $AnnM$ where M is an irreducible torsion free module. This could be viewed as an approximation to the classification of irreducible torsion free modules. We now use the set, $wt\ M$, of weights of M to define a representation $\rho_M : \pi_1(\mathcal{H}^* \setminus D) \longrightarrow GL(n,\mathbb{C})$ of a certain hyperplane complement $\mathcal{H}^* \setminus D$ in $\mathcal{H}^*$. It turns out that torsion free irreducible modules can be classified by the data $(AnnM, \rho_M)$. The inverse of the map $M \longrightarrow (AnnM, \rho_M)$ is described by using an explicit geometric construction. To show that the construction does in fact yield M, first we reduce to the case where $\mathcal{U}(\mathcal{G})/AnnM$ is a ring of differential operators. Here we use a translation principle and the classification of the annihilators of irreducible torsion free modules. Then in the special case where $\mathcal{U}(\mathcal{G})/AnnM$ is a ring of differential operators, we observe that the map $M \longrightarrow \rho_M$ is bijective because it amounts essentially (i.e. after passing to an associated D-module) to a Riemann-Hilbert correspondence.

REFERENCES

[Fe1] S. Fernando, Lie algebra modules with finite dimensional weight spaces I, to appear.

[Fe2] S. Fernando, Lie algebra modules with finite dimensional weight spaces II, to appear.

VIRASORO ALGEBRA AND COSET CONSTRUCTIONS

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A review was given of the coset construction of representations of the Virasoro algebra $\widehat{v}$. Unitary irreducible highest weight representations of $\widehat{v}$ are labelled by $(c, h) \in \mathbb{R}^2$, with either $c \geq 1, h \geq 0$ (continuum) or $c = 1 - \frac{6}{(m+1)(m+2)}$, $m = 0, 1, 2, \ldots$ and h taking one of $\frac{1}{2}(m+1)(m+2)$ values (discrete series). The *Sugawara construction* of the Virasoro algebra starts with an irreducible unitary highest weight representation of an affine algebra $\widehat{\mathcal{G}}$ defined by $[T^a_m, T^b_n] = if^{abc}T^c_{m+n} + km\delta^{ab}\delta_{m,-n}$ and sets $L^{\mathcal{G}}_n = \frac{1}{\beta}(\sum_m {}^{\times}_{\times} T^a_m T^a_{n-m} {}^{\times}_{\times})$ where $\beta = 2k + Q^{\mathcal{G}}, Q^{\mathcal{G}}$ being the quadratic Casimir operator of $\mathcal{G}$ in the adjoint representation and ${}^{\times}_{\times} \; {}^{\times}_{\times}$ denotes the normal ordering. This has $c = c^{\mathcal{G}} = 2k\, dim\mathcal{G}/(2k + Q^{\mathcal{G}})$ and $rank\,\mathcal{G} \leq c^{\mathcal{G}} \leq dim\,\mathcal{G}$. The $coset - construction$ proceeds by considering a pair $\mathcal{G} \supset \mathcal{H}$ and consequently $\widehat{\mathcal{G}} \supset \widehat{\mathcal{H}}$, setting $K_n = L^{\mathcal{G}}_n - L^{\mathcal{H}}_n$ so defining a Virasoro representation with $c = c_K = c^{\mathcal{G}} - c^{\mathcal{H}}$. Taking $\mathcal{G} = \widehat{su}(2)_m \times \widehat{su}(2)_1$, and $\mathcal{H} = \widehat{su}(2)_{m+1}$, the diagonal subalgebra, the suffixes denoting levels, provides all the discrete series representations.

The construction enables information about Virasoro representations to be deduced from those of the affine algebras, in particular the modular properties of characters. If $\mathcal{G} \supset \mathcal{H}$ we can consider the decomposition of representations of $\widehat{\mathcal{G}}$ at a particular level with respect to those of the direct product of $\widehat{\mathcal{H}}$ (at the reduced level) and the Virasoro algebra $v_K \equiv \{K_n, c_K\}$. This decomposition is finite if and only if $c_K < 1$ and the decomposition of $\widehat{\mathcal{G}}$ with respect to $\widehat{\mathcal{H}}$ is finite if and only if $c_K = 0$. In the latter case we can use sesquilinear modular invariant combinations of characters of $\widehat{\mathcal{G}}$ to construct sesquilinear combinations for $\widehat{\mathcal{H}}$ and, in the former, we can use such combinations for $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{H}}$ to construct them for $\widehat{v}$.

Extensions of those results obtain for the super-Virasoro algebras and for more general extensions of $\widehat{v}$ which are not Lie algebras but are defined by operator product expansions.

ON KAC'S "RECOGNITION THEOREM"

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As part of a program for classifying the simple Lie algebras of prime characteristic which he outlined in the early seventies, Victor Kac proved the theorem which Richard Block and Robert Wilson refer to as Kac's "Recognition Theorem" for graded Lie algebras, and which they made use of in their recent classification of the restricted simple finite dimensional Lie algebras over algebraically closed fields of prime characteristic. It is expected that the approach used in their proof can be generalized to the non-restricted case. One potentially useful ingredient in such a generalization would be a "Recognition Theorem" without the hypothesis that the adjoint representation of the null component G_0 on the minus-one component G_{-1} of the graded algebra G be restricted. Georgia Benkart and I were able to show that Kac's theorem remains true without this hypothesis. One notion used in the proof of our Main Theorem is that of the character of a representation of a restricted Lie algebra. The character χ of the representation of G_0 on G_{-1} is zero if the representation is restricted, so our main objective is to prove that χ is zero. Our method of showing that χ disappears on a particular G_{-t} involves the construction of certain depth-one quotient Lie algebras, which we denote $B(t)$. If the one-component of $B(t)$ is not zero, we can show that the representation of the zero component on the minus-one component of $B(t)$ is restricted. That implies that the representation of G_0 on G_{-t} is restricted. We are able to show that the one-component of $B(t)$ is not zero when t is equal to $q-i$, where q is the depth of G and $i = 0, 1, 2$, or 3. In this way, we are able to prove the Main Theorem for all primes greater than 5.

VERMA BASES FOR HIGHEST WEIGHT MODULES

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Let $\mathcal{G}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ of rank ℓ, let $\mathcal{H}$ be a Cartan subalgebra, and let $e_1, \ldots, e_\ell, f_1, \ldots, f_\ell$ be Chevalley generators. Let $\alpha_1, \ldots, \alpha_\ell$ denote the simple roots, $r_i = r_{\alpha_i}$ the reflection in α_i, and $\mathcal{W}$ the Weyl group of $\mathcal{G}$.

For $V = V(\lambda)$ a finite-dimensional irreducible $\mathcal{G}$-module of highest weight $\lambda \in \mathcal{H}^*$, D-N. Verma has proposed the following method for obtaining a basis for V. Let v^+ be a highest weight vector for V and $r_{i_n} \cdots r_{i_2} r_{i_1}$ a reduced expression for w_0, the unique element of maximum length in $\mathcal{W}$. Verma's proposal consists primarily of an algorithm for finding certain functions $U_1, \ldots, U_n$ so that the set of all elements of the form

$$f_{i_n}^{a_n} \cdots f_{i_2}^{a_2} f_{i_1}^{a_1} \cdot v^+$$

such that

$$\begin{aligned} &0 \le a_1 \le U_1 \\ &0 \le a_2 \le U_2(a_1) \\ &\vdots \\ &0 \le a_n \le U_n(a_1, \ldots, a_{n-1}) \end{aligned}$$

should be a basis for V.

This algorithm does not work for all reduced expressions for w_0. To date, "good" reduced expressions have been found for the classical algebras and for G_2; no one has had any luck yet with E_6, E_7, E_8 or F_4. Furthermore, even when the algorithm does work, it is by no means obvious, that the resulting elements form a basis. That a basis is obtained has only been shown for the algebras A_ℓ, B_2 and G_2, although some progress has been made on B_ℓ ($\ell \geq 3$), C_ℓ and D_ℓ. Finally, most of these results and ideas can be extended to Verma modules and to arbitrary Kac-Moody algebras.

ON RATIONALITY PROPERTIES OF INVOLUTIONS OF REDUCTIVE GROUPS

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Let k be a field of characteristic not two and $\mathcal{G}$ a connected linear reductive k-group. Write $\mathcal{G}_k$ for the set of k-rational points of $\mathcal{G}$. By a k-involution Θ of $\mathcal{G}$, we mean a k-automorphism Θ of $\mathcal{G}$ of order two. Let $\mathcal{H} = \mathcal{G}_\Theta$ be the fixed point group of Θ. For $k = \mathbb{R}$ or an algebraically closed field, such involutions have been extensively studied emerging from different interests. Especially the interactions with the representation theory of reductive groups have been most rewarding. In this talk we present a survey on rationality problems of general k-involutions; this with an emphasis on a characterization of the double coset space $\mathcal{P}_k \setminus \mathcal{G}_k / \mathcal{H}_k$ where $\mathcal{P}$ is a minimal parabolic k-subgroup of $\mathcal{G}$. The geometry of these orbits is of importance for representation theory. We also discuss the orbit closures and dimension formulas. We also generalize the notion of Cartan involution for $k = \mathbb{R}$ to a more general setting. This leads among others to a more precise description of the double cosets $\mathcal{P}_k \setminus \mathcal{G}_k / \mathcal{H}_k$.

MODULAR INVARIANT REPRESENTATIONS

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In this talk I discuss the problem of classification of modular invariant representations of affine like and Virasoro like Lie algebras and superalgebras. The key result is a character formula for a large class of highest weight representations of a Kac-Moody algebra and superalgebra, generalizing the Weyl-Kac character formula. In the case of affine algebras, this class includes modular invariant representations of arbitrary rational level m such that $(m+g) \geq g/u$, where $u > 0$ is the denominator of m and g is the dual Coxeter number. In the case of $A_1^{(1)}$ this gives a complete classification of modular invariant representations. I discuss also in detail the modular invariant representations of the Virasoro algebra, their connection to modular invariant representations of $A_1^{(1)}$ and to the Rogers-Ramanujan identities.