CISM COURSES AND LECTURES NO. 223
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

# THERMOMECHANICS IN SOLIDS

EDITED BY

W. NOWACKI / I.N. SNEDDON

PRINGER - VERLAG
VIEN NEW YORK

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# THERMOMECHANICS IN SOLIDS

A SYMPOSIUM
HELD AT CISM, UDINE, IN JULY 1974

EDITED BY

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SPRINGER - VERLAG WIEN - NEW YORK

ISBN 3-211-81343-8 Springer - Verlag Wien - New York
ISBN 0-387-81343-8 Springer - Verlag New York - Wien

# PREFACE

*In the past, many fields of Mechanics, including that of Mechanics of Continuous Media, have developed more or less independently of Thermodynamics. At present, however, a stage has been reached where further progress appears impossible without the inclusion of thermodynamic concepts.*

*In these circumstances, the International Centre for Mechanical Sciences (CISM) decided to present and discuss the results obtained in two course series which were held in 1971 and 1972. In the first one emphasis was placed on the basic concepts and their applications. In particular, the classical approach to thermodynamics of irreversible processes as well as the modern relevant concepts were presented and discussed. In the second series, the developments of the theory of thermal stresses were reviewed and its applications in the field of mechanical, air and spacecraft engineering were surveyed. Besides, attention was focussed on coupled thermoelasticity developed as a synthesis of the theory of elasticity and the theory of thermal conduction. Basic research problems, dispersion of elastic waves, dissipation of energy, etc. were dealt with. Stationary and non-stationary problems, both in the frame of a linear approach and for finite deformations, were analyzed. The lectures were also devoted to thermal disturbances in bodies of various physical response, to thermal effects in piezoelectric media, to the fundamentals of magneto-thermoelasticity and to problems of thermodiffusion in solids.*

*These course series turned out to be a real success, and also stimulated, creative activities in various research centres. As a consequence, the participants suggested to meet again in about two or three years in order to get acquainted with the new trends of development of thermomechanics.*

*As a result, a Symposium on "Thermoelasticity" has been held at the CISM in Udine during its Rankine Session, from July 22 to 25, 1974. It has been organized with the aim to survey the steadily growing achievements in this area as well as to discuss questions of further progress.*

*In order to initiate the discussion, four general reports have been invited. These reports are contained in the present volume. They were scheduled to be read at four consecutive days whereupon the original contributions and papers in the corresponding special fields were delivered. These (fifteen) contributions have subsequently been published in various Journals.*

*In the first part of this volume Professor I.N. Sneddon presents and* *discusses the coupled problems of linear thermoelasticity whilst its generalizations and developments in the field of anisotropic elastic media are dealt with, in the second part, by Professor W. Nowacki.*

*Since in recent years there has been a rapid development in the phenomenological theory of coupled electromagnetic and deformation fields, Professor H. Parkus extends this work to include thermo-magneto-elastic interactions.*

*Finally, professor C. Woźniak presents a new area of development, the thermoelasticity of nonlinear discrete and continuum constrained systems.*

*During the Symposium considerable time was devoted to informal discussions, and the Symposium was concluded by a round-table discussion in which the main features of progress in Thermomechanics of Solids have been critically reviewed and treated. The participants agreed on the need of further meetings of a similar kind.*

*W. Nowacki* *W. Olszak.*

*Udine, December 1977.*

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# COUPLED PROBLEMS IN THE LINEAR THEORY OF THERMOELASTICITY

**Ian N. Sneddon**

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## 1 Introduction

The purpose of this introductory lecture is to present a brief account of the linear theory of thermoelasticity starting with a discussion of the basic equations of (non-linear) thermoelasticity and deriving the coupled equations of linear thermoelasticity from them. The treatment leans heavily on the articles [1] and [2].

This is followed by a discussion of the mixed problem of the dynamical theory of linear thermoelasticity and of the variational principles which may be used to derive solutions in special cases.

The next two sections are concerned, respectively, with the propagation of harmonic plane waves in a homogeneous isotropic elastic solid and with a description of some special solutions of the coupled equations.

The survey ends with some remarks on problems with finite wave speed in the heat conduction equation.

## 2. The basic equations of thermoelasticity

The elastic body $B$ is identified with the bounded regular region of space it occupies in a fixed reference configuration $\mathscr{C}$. The displacement at time $t$ of the point $x \in B$ is denoted by $\mathbf{u}(x,t)$. We shall suppose that $t$ belongs to the finite time interval $(0,t_0)$ and we shall write

$$\Omega = B \times (0,t_0), \quad \bar{\Omega} = \bar{B} \times [0,t_0],$$

where $\bar{B}$ denotes the closure of $B$. By a *motion* of the body we mean a vector field $u \in C^2(\Omega)$ and by the *deformation gradient* the spatial gradient F of the mapping which takes the point x to $x+u(x,t)$, i.e.

$$\mathbf{F} = 1 + \nabla\mathbf{u} \tag{2.1}$$

where 1 denotes the unit tensor and $\nabla u$ the gradient of u. It is assumed that the motions under consideration are such that the mapping $\mathbf{x} \mapsto \mathbf{x}+\mathbf{u}(x,t)$ is injective on $B$ and has a smooth inverse so that $\det \mathbf{F} \neq 0$.

If we denote the *Piola-Kirchhoff stress tensor* (measured per unit surface area in the reference configuration) by $\mathbf{S}(x,t)$ and the *body force* (measured per unit volume in $\mathscr{C}$) by $\mathbf{f}(x,t)$ the laws of balance of forces and moments lead to the equations

$$\operatorname{div} \mathbf{S} + \mathbf{f} = 0, \tag{2.2}$$

$$\mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T, \tag{2.3}$$

where $F^T$ denotes the transpose of the tensor F.

If we denote the *internal energy* (per unit volume in $\mathscr{C}$) by $e(x,t)$, the heat flux vector by $q(x,t)$ and the heat supply, per unit volume in $\mathscr{C}$, by $r(x,t)$ and if we assume that $e \in C^{0,1}(\Omega)$, $q \in C^{1,0}(\Omega)$, $r \in C(\Omega)$ then the local form of the first law of thermodynamics is expressed by the equation

(2.4) $$\dot{e} = S.\dot{F} - \operatorname{div} q + r.$$

Similarly, if $\eta(x,t) \in C^{0,1}(\Omega)$ is the *entropy*, $\theta(x,t) \in C^{1,0}(\Omega)$ is the *absolute temperature* with $\theta(x,t) > 0$, the local form of the second law of thermodynamics is expressed by the inequality

(2.5) $$\dot{\eta} \geqslant -\operatorname{div}(q/\theta) + r/\theta$$

If we introduce the free energy $\psi = e - \eta\theta$ and the temperature gradient $g = \nabla\theta$ we can write this last inequality in the alternative form

(2.6) $$\dot{\psi} + \eta\dot{\theta} - S.\dot{F} + (g.\dot{q})/\theta \leqslant 0;$$

the inequality (2.6) is called the *local dissipation inequality..*

So far we have made no assumptions concerning the nature of the material forming the body B . Now we assume that the material is *elastic,* that is, that there exist four *constitutive equations* which define $\psi$, S , $\eta$ and q as smooth functions of the set of all $(F,\theta,g,x)$ for which div $F \neq 0$ and $\theta > 0$ . Certain restrictions are imposed on these constitutive equations by the local dissipation inequality, that is, by the second law of thermodynamics. (See, e.g. [2], [4], [5]). It turns out that $\psi$, S and $\eta$ are independent of the temperature gradient g and that q satisfies the relation

(2.7) $$(g.q) \leqslant 0.$$

This last inequality is called *the heat conduction inequality.*

Further, if, for convenience, we omit the variable x and write

(2.8) $$S = \hat{S}(F,\theta), \quad \psi = \hat{\psi}(F,\theta), \quad \eta = \hat{\eta}(F,\theta),$$

we find that $\hat{S}$ and $\hat{\eta}$ can be calculated from $\hat{\psi}$ by means of the

(2.9) $$\hat{S}(F,\theta) = \partial_F\hat{\psi}(F,\theta), \quad \hat{\eta}(F,\theta) = -\partial_\theta\hat{\psi}(F,\theta).$$

The first of these equations is called the *stress relation,* the second is called the *entropy relation* and the equation

(2.10) $$\partial_\theta\hat{S}(F,\theta) = -\partial_F\hat{\eta}(F,\theta)$$

found by eliminating $\hat{\psi}$ between them is called *Maxwell's relation.*

From the definition of the free energy $\psi$ we deduce immediately that the internal energy obeys a constitutive relation $e = \hat{e}(F.\theta)$ where the function $\hat{e}$ is defined by the equation

$$\hat{e}(F,\theta) = \hat{\psi}(F,\theta) + \theta\hat{\eta}(F,\theta). \tag{2.11}$$

Defining the specific heat $c(F,\theta)$ of the material through the equation

$$c(F,\theta) = \partial_\theta \hat{e}(F,\theta) \tag{2.12}$$

we deduce immediately from equations (2.11) and $(2.9)_2$ that

$$c(F,\theta) = \theta\partial_\theta \hat{\eta}(F,\theta) \tag{2.13}$$

We shall confine our attention to materials for which the specific heat is strictly positive, and since, by hypothesis, $\theta > 0$ we deduce immediately from equations (2.12) and (2.13) that the function $\hat{\eta}(F,\theta)$ has a smooth inverse in $\theta$ for each choice of F, i.e. that we may write $\theta = \bar{\theta}(F,\eta)$ and hence that we may write the consititutive equations in the alternative forms

$$e = \bar{e}(F,\eta), \quad S = \bar{S}(F,\eta), \quad \theta = \bar{\theta}(F,\eta), \quad q = \bar{q}(F,\eta) \tag{2.14}$$

The relations (2.9) then imply the pair of equations

$$\bar{S}(F,\eta) = \partial_F \bar{e}(F,\eta), \qquad \bar{\theta}(F,\eta) = \partial_\eta \bar{e}(F,\eta) \tag{2.15}$$

Further conditions on the constitutive equations of an elastic material are obtained by applying the *principle of material frame indifference* (sects. 17-19A of [6]) which states that the constitutive equations are independent of the observer. For this to be so the consititutive equations must have the *reduced forms*

$$\psi = \tilde{\psi}(D,\theta), \quad S = F\tilde{S}(D,\theta), \quad \eta = \tilde{\eta}(D,\theta), \quad q = \tilde{q}(D,\theta,g) \tag{2.16}$$

where

$$D = \frac{1}{2}(F^T F - 1) \tag{2.17}$$

is the *finite strain tensor*, and $\tilde{\psi}$, $\tilde{S}$ and $\tilde{\eta}$ satisfy the equations

$$\tilde{S}(D,\theta) = \partial_D \tilde{\psi}(D,\theta), \qquad \tilde{\eta}(D,\theta) = -\partial_\theta \tilde{\psi}(D,\theta) \tag{2.18}$$

The heat conduction inequality (2.7) also has important consequences. If we define the *conductivity tensor* $K(D,\theta)$ by the equation

(2.19) $$K(D,\theta) = - \partial_g \tilde{q}(D,\theta,0)$$

then as a consequence of (2.7) we have that $K(D,\theta)$ is positive semi-definite, and that

(2.20) $$\tilde{q}(D,\theta,0) = 0, \quad \partial_D \tilde{q}(D,\theta,0) = 0, \quad \partial_\theta q(D,\theta,0) = 0$$

## 3. The linear theory of thermoelasticity

We now consider the linear approximation to the system of equations of thermoelasticity consequent upon the following assumptions:-

(a) the displacement gradient $\nabla \mathbf{u}$ and its time rate of change $\nabla \dot{\mathbf{u}}$ are both small;

(b) the temperature field differs only slightly from a prescribed, *uniform* temperature field $\theta_0$, called the *reference temperature;* i.e. $|\vartheta/\theta_0| \ll 1$ where $\vartheta = \theta - \theta_0$;

(c) the time rate of change of the temperature, $\dot{\theta}$, and the temperature gradient $\mathbf{g}$ are small.

If $|\nabla \mathbf{u}| \leq \delta_1$, then it is easily that $\mathbf{D} = \mathbf{E} + O(\delta_1^2)$ as $\delta_1 \to 0$, where $\mathbf{E}$ is the *infinitesimal strain tensor* defined by the equation

(3.1) $$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T),$$

and similarly if $|\nabla \dot{\mathbf{u}}| \leq \delta_2$, $\dot{\mathbf{D}} = \dot{\mathbf{E}} + O(\delta_2^2)$ as $\delta_2 \to 0$. Also, if $|\vartheta/\theta_0| \leq \delta_3$ and $\delta = \max(\delta_1, \delta_3)$ we find that

(3.2) $$\tilde{\mathbf{S}}(D,\theta) = \mathbb{C}[\mathbf{E}] + (\theta - \theta_0)\mathbf{M} + O(\delta), \quad \delta \to 0$$

where $\mathbf{C}$ and $\mathbf{M}$ are defined by the equations

(3.3) $$\mathbf{C} = \partial_D \tilde{\mathbf{S}}(0,\theta_0) = \partial_D^2 \tilde{\psi}(0,\theta_0),$$

(3.4) $$\mathbf{M} = \partial_\theta \tilde{\mathbf{S}}(0,\theta_0) = \partial_\theta \partial_D \tilde{\psi}(0,\theta_0),$$

respectively; the fourth order tensor $\mathbf{C}$ is called the *elasticity tensor* and the second order symmetric tensor $\mathbf{M}$ is called the *stress temperature tensor*. It should also be noted that, for any pair of symmetric tensors G and H

(3.5) $$G.\mathbf{C}[H] = H.\mathbf{C}[G];$$

in component form this is equivalent to the symmetry condition

$$C_{k\ell ij} = C_{ijk\ell} \tag{3.6}$$

In a similar way we obtain the approximation

$$q = -Kg \tag{3.7}$$

where

$$K = -\partial_g \tilde{q}(0, \theta_0, 0) \tag{3.8}$$

is the *conductivity tensor.* It should be emphasized that there is no reason to believe that, in general, K will be a symmetric tensor; it is always positive semi-definite.

Denoting the density by $\rho(x)$ and the non-inertial body force by b so that $f = b - \rho\ddot{u}$ we see that equation (2.2) becomes

$$\text{div } S + b = \rho\ddot{u} \tag{3.9}$$

Finally, if we introduce the scalar

$$c = \theta_0 \partial_\theta \tilde{\eta}(0, \theta_0) \tag{3.10}$$

– *the specific heat* corresponding to $D = 0$ and $\theta = \theta_0$ - we obtain

$$-\text{div } q + \theta_0 M.\dot{E} + r = c\dot{\theta} \tag{3.11}$$

as the linearized form of the energy equation (2.5).

Collecting these equations together we have :–

*The basic equations of the linear theory of thermoelasticity:–*

$$\left.\begin{aligned} E &= \frac{1}{2}(\nabla u + \nabla u^T), \\ \text{div } S + b &= \rho\ddot{u}, \\ -\text{div } q + \theta_0 M.\dot{E} + r &= c\dot{\theta} \\ S &= C[E] + (\theta - \theta_0)M \\ q &= -K\nabla\theta \end{aligned}\right\} \tag{3.12}$$

**4. The linear theory in the isotropic case**

The form of the tensors **C**,**M** and **K** are particularly simple in the isotropic case. (See Sect. 21, 22 and 26 of [1] and [7]). We have

$$C[E] = 2\mu E + \lambda(\mathrm{tr}\,E)1$$

$$\mathbf{M} = m1 \tag{4.1}$$

$$\mathbf{K} = k1$$

where $\lambda$ and $\mu$ are the Lamé constants, k is the conductivity and, in terms of $\alpha$, the coefficients of thermal expansion

$$m = -(3\lambda + 2\mu)\alpha . \tag{4.2}$$

We therefore have

*The basic equations of linear thermoelasticity for an isotropic body :–*

$$\left.\begin{aligned} E &= \frac{1}{2}(\nabla u + \nabla u^T), \\ \mathrm{div}\,S + b &= \rho\ddot{u} \\ -\mathrm{div}\,q + m\Theta_0 \mathrm{tr}\dot{E} + r &= c\dot{\theta}, \\ S &= 2\mu E + \{\lambda(\mathrm{tr}E) + m\vartheta\}1 \\ q &= -k\nabla\theta \end{aligned}\right\} \tag{4.3}$$

The first two and the last two equations of this system were first derived by Duhamel [8] and later by Neumann [9].
In both cases the strain-rate term

$$m\theta_0 \;\mathrm{tr}\dot{E}$$

did not appear in the third equation of the set representing the energy balance. There were attempts, at a later date, to justify the inclusion of such a term on the basis of reversible thermodynamics by Voight [10], Jeffreys [11] and Lessen and Duke [12], and on the basis of irreversible thermodynamics by Biot [13]. The derivation outlined here is that given in Sect. 3-8 of [2]; a similar treatment based on modern continuum mechanics and thermodynamics is given in Chap. 8 of Eringen's book [14].

In many applications of the theory two additional assumptions are often made to facilitate the solution of boundary value problems. The first of these – which leads to the *uncoupled theory* – is to assume that in the energy balance equation the term $\theta_0 \operatorname{tr}\dot{E}$ may be neglected so that the temperature field is determined by the pair of equations

$$-\operatorname{div} q + r = c\dot{\theta}, \quad q = -k\nabla\theta .$$

Once the temperature field has been calculated the stress and displacement fields can then be found by use of the first, second and fourth equations of the system (4.3). The second simplifying assumption – which leads to the *quasi-static theory* – is that the inertia term $\rho\ddot{u}$ in the second equation of the system (4.3) may be neglected but that the equations are otherwise unaltered. Indeed, in many engineering applications, in which the geometry is complicated, both approximations are made simultaneously. Such approximate solutions are discussed in the books by Melan and Parkus [15], Boley and Weiner [16], Nowacki [17] and Kovalenko [18].

Here, we shall continue with the discussion of properties of the full set of coupled equations.

If we eliminate S and E from the first, second and fourth of these equations we obtain the equations of motion

$$\mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u + m\nabla\vartheta + b = \rho\ddot{u}, \tag{4.4}$$

while if we eliminate q and E from the first, third and fifth equations of the set we obtain the coupled heat equation

$$k\Delta\vartheta + m\theta_0 \operatorname{div}\dot{u} + r = c\dot{\vartheta} \tag{4.5}$$

Applying the operator div to both sides of equation (4.4), and the operator curl to both sides of equation (4.5) we obtain the inhomogeneous wave equations

$$\Box_1 \operatorname{div} u = -\rho^{-1}(m\Delta\vartheta + \operatorname{div} b), \tag{4.6}$$

$$\Box_2 \operatorname{curl} u = -\rho^{-1} \operatorname{curl} b, \tag{4.7}$$

in which the operators $\Box_1$, $\Box_2$ are defined by the equations

$$\Box_\alpha f = c_\alpha^2 \Delta f - \ddot{f}, \quad (\alpha = 1,2) \tag{4.8}$$

with

$$c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho \tag{4.9}$$

so that $c_1$ and $c_2$ are respectively the velocities of the P– and the S– waves in