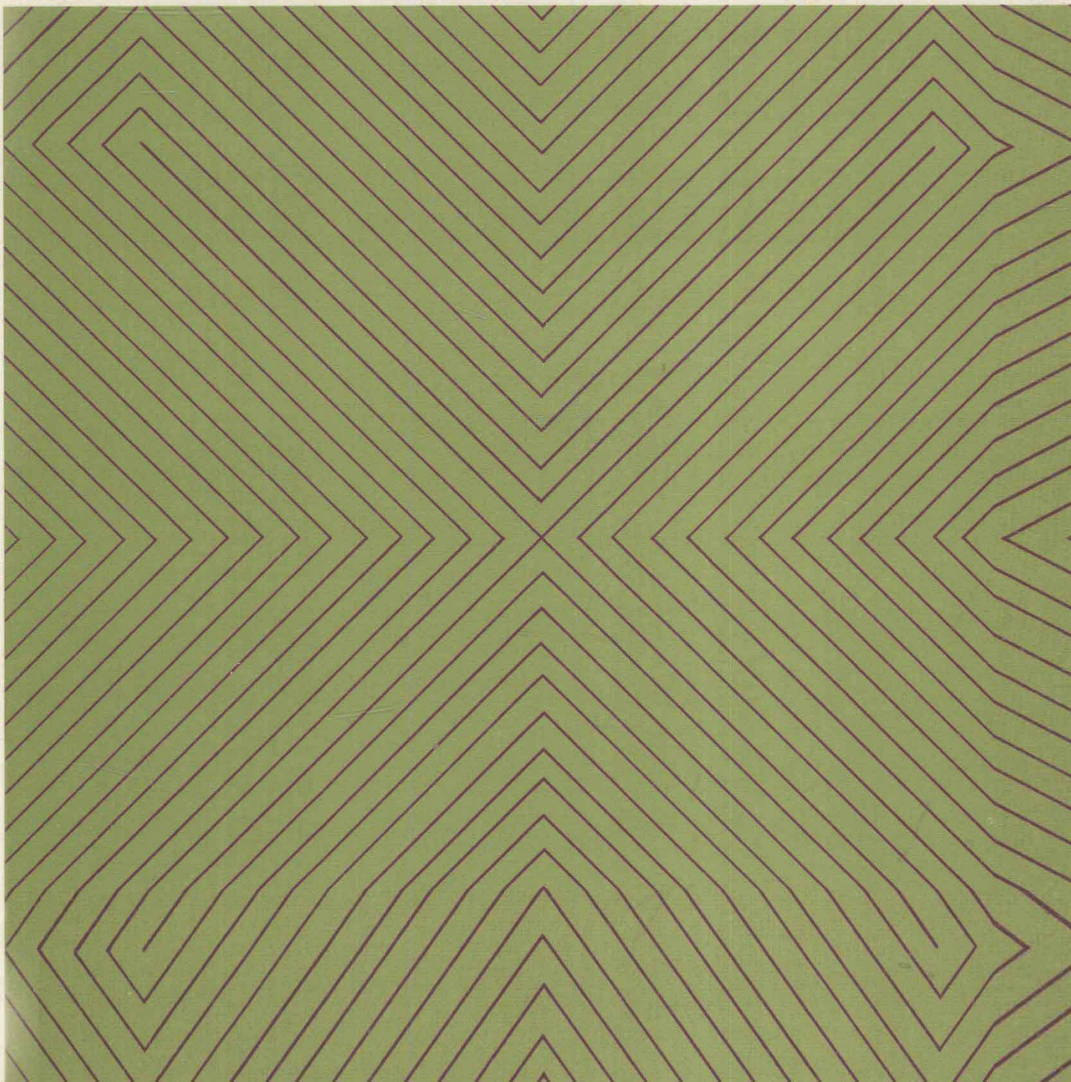
John de Pillis

# LINEAR ALGEBRA



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**Holt, Rinehart and Winston, Inc.**
NEW YORK / CHICAGO / SAN FRANCISCO / ATLANTA
DALLAS / MONTREAL / TORONTO / LONDON / SYDNEY

Library of Congress Catalog Card Number: 69–14251
**SBN: 03–073595–5**
Printed in the United States of America
1 2 3 4 5 6 7 8 9

# Linear Algebra

*In memory of my mother*
***Lili de Pillis***

# Preface

**Description.** Suitable for freshman or sophomore students, this introduction to linear algebra begins with a discussion of concrete vector spaces after which systems of linear equations are encountered. Vector-space notions are immediately applied in presenting a theory of linear programming in the plane. We then proceed to linear transformations and their matrices, followed by a deeper study of linear equations. Next, determinants are studied from both a classical and nonclassical approach. The final chapter deals with the structure of operators and introduces ideas of inner product, diagonalization, and the spectral decomposition theorems for self-adjoint operators and Hermitian matrices. Each chapter contains examples, exercises, and a summary of the material discussed.

A complete chart of all theorems and definitions of the book is followed by a self-contained appendix on mathematical induction and an appendix on complex numbers.

**Features.** We are guided by two general principles: First, new ideas should be offered in a concrete (or particular) setting and should then evolve into the more abstract (or general) situation. Second, sufficient space should be given to establishing motivation and anticipating the student's questions. (Why is this definition necessary? Where did it come from? Why is this theorem given in this form?)

Consistent with these general aims, the following particular features are incorporated in this text:

- Concrete $n$-vectors are introduced at the outset and vector-space notions are immediately used to develop a theory of linear programming in the plane. As the first chapter unfolds, we are systematically led to the definition of an *abstract* vector space.
- Systems of simultaneous linear equations are briefly explored in Chapter I

where the technique of Gaussian Elimination is described. The justification of this technique, along with a deeper study of linear systems, appears later in Chapter III.

- Each chapter contains an abundance of exercises and illustrative examples.
- By use of the notation $mx(\mathbf{T})_{\mathscr{X}\mathscr{Y}}$ for the matrix of a linear transformation **T**, we have tried to underscore the fact that the matrix depends on the *three* entities which are: the linear transformation **T**, the ordered basis $\mathscr{X}$ of the domain space, and the ordered basis $\mathscr{Y}$ of the range space. Many examples and exercises dwell on this dependence, as well.
- Matrices are discussed for the $2 \times 2$ case first. This special case may be omitted, if desired, in favor of the general $m \times n$ case which follows.
- Determinants are presented in two ways. First, in Chapter IV, we have the classical development depending on properties of permutation functions. Alternately, Chapter V offers the nonclassical multilinear viewpoint which does not require permutations.
- The spectral theorem for self-adjoint operators and hermitian matrices is not presented as a dead-end finale. This important decomposition finds extensive use at the end of Chapter VI.
- Important computational techniques of the text are immediately retrievable via the unique "computations" entry in the index. For example, five references are given for techniques of finding the inverse of a matrix under the "matrix inverse" subheading of the "Computations" entry.
- For easy location of theorems and definitions, we use a consecutive triple entry numbering system which appears in the running head of every page. As an example, item (theorem or definition) II.3.9 appears in Chapter II, Section 3, and is the ninth item of that section.
- Each chapter concludes with a summary.
- A bird's-eye view of the entire text appears for the student's convenience in the form of a chart of all theorems and definitions (following Chapter VI).

**Acknowledgments.** Most importantly, it was my wife Susie who encouraged me in the writing of this book. I am grateful for the helpful comments and suggestions of Professors R. M. Thrall and J. Murtha, who read the entire manuscript in raw form. Also, much credit is due Mrs. Ollie Cullers, Mrs. Jane Scully, and Mrs. Sharon Unangst, who translated scribbled notes into a polished typed form. For the use of their facilities, I wish to thank the University of California at Riverside, and Brookhaven National Laboratory, Upton, New York.

*Riverside, California* JOHN DE PILLIS
*January 1969*

# Contents

# Introduction

The purpose of this introduction is to set down the language of set theory. The definitions presented are somewhat inexact in that the undefined terms are not explicitly cited. The reason for this omission is that a rigorous development of the theory of sets would require techniques that are quite advanced. However, it is hoped that the concepts will sit well with the readers' intuition.

***Definition 1*** A **set** will denote a collection (family, aggregate) of distinguishable objects, which will be called **elements** or **members** of the set.

***Notation 2*** If $A$ represents a set, and $x$ represents one of its elements (members), then we use the notation

$$x \in A, \quad \text{read} \quad \text{"}x \text{ belongs to the set } A,\text{"}$$

or "$x$ is an element of $A$," or "$x$ is a member of $A$." Not too surprisingly, then, we use the notation $x \notin A$ to stand for the negative of the previous statement(s), that is, "$x$ is not an element of $A$," and so forth.

***Notation 3*** Suppose the symbols $x_1, x_2, \ldots, x_n, \ldots$ represent a "listing" of the elements of a set $A$. (Here, the dots ... are to stand for the phrase "and so on" which indicates that we really know what all the elements of $A$ are (somehow), but use this shorthand as a convenience). In this case, we could simply write

$$A = \{x_1, x_2, \ldots, x_n, \ldots\}.$$

**Example 1** Let $A$ be the 4-element set of letters $y$, $a$, $r$, $g$. That is (Notation 3), $A = \{y,a,r,g\}$. Alternatively, $A = \{g,r,a,y\}$ or $A = \{r,a,y,g\}$. If you like to think of $A$ as the word "yarg," then you are artificially imposing an *ordering* on the elements of $A$. Heretofore, we have mentioned nothing of ordered sets, although they will prove useful to us later.

**Example 2** Let $A$ represent the 3-element set whose elements are $\{y,a,r,g\}$, $\{p,o,t\}$, and $\{n,u,t,s\}$. Notice that the elements are themselves *sets*. Thus, $A = \{\{y,a,r,g\}, \{p,o,t\}, \{n,u,t,s\}\}$. According to Notation 2, we make the statement,

$$\{p,o,t\} \in A,$$

or

$$\{p,o,t\} \in \{\{y,a,r,g\}, \{p,o,t\}, \{n,u,t,s\}\}.$$

Now, each *letter* is *not* an element of $A$. That is,

$$p \notin A;$$

rather, $p \in \{p,o,t\}$, and the set $\{p,o,t\}$ is an element of $A$.

## EXERCISES

1 Let $A$ be the set whose elements are the integers from 3 to 7 inclusive.
   (a) Is it true that $4 \in A$ ?
   (b) Is it true that $\{3,4\} \in A$ ?
   (c) Is it true that $2 \in A$?

2 Let $A$ be the following set of *words* (here, a *word* is to be interpreted as an *ordered set* in itself, for example $\{c,a,t\}$ is a word, but is not the same word as $\{t,a,c\}$ since the *orderings* of the letters differ):

$$A = \{\{c,a,t\}, \{t,a,c\}, \{f,r,o,g\}, \{g,a,r,f\}\}.$$

   Which of the following statements is true? Explain.
   (a) $\{f,r,o,g\} \in A$
   (b) $f \in \{f,r,o,g\}$
   (c) $f \in A$
   (d) $\{f,r,o,g,a,r,f\} \in A$
   (e) $\{t,a\} \in A$.

***Definition 4*** Let $A$ and $B$ be sets. Then the symbols $A \subset B$ (equivalently, $B \supset A$), read **$A$ is contained in $B$**, **$B$ contains $A$** or **$A$ is a subset of $B$**, is to mean that each and every element of $A$ is also an element of $B$. That is to say, $A \subset B$ means:

$$\text{whenever} \quad x \in A, \quad \text{then} \quad x \in B.$$

**Example 3** Let $A = \{1,2\ 3\ 4\}$ and $B = \{1,3\}$. We assert that $B \subset A$, since (as Definition 4 requires), every element of $B$ is also an element of $A$. The elements of $B$ are exactly the integers 1 and 3 and surely, $1 \in \{1,2,3,4\}$ and $3 \in \{1,2,3,4\} = A$.

With the notion of inclusion (Definition 4) in hand, we are able to say precisely what is meant in saying that one set $A$ is "equal to" another set $B$. Informally, we shall write $A = B$ whenever elements of $A$ are always elements of $B$, and *vice versa*. Formally, we have Definition 5.

***Definition 5*** Let $A$ and $B$ be sets. The symbol $A = B$, read **$A$ equals $B$** or **$B$ equals $A$**, is to mean that $A \subset B$ *and* $B \subset A$.

**Example 4** Let $A$ be the set of all even integers, that is, integers which are of the form $2 \cdot q$ where $q$ is an arbitrary integer. We construct the set $B$ as follows: Select the integer 0. Then leave the adjacent integers alone (in this case leave 1 and $-1$ alone) and select the "next" integers (in this case, select 2 and $-2$). Repeat this selection process of selecting "every other" integer. The set of all selected integers (elements) will be our set $B$.

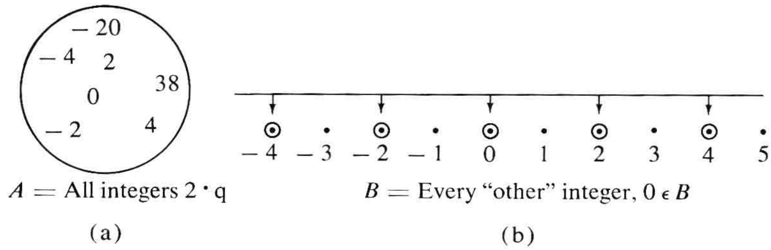



***Figure 1***

Note that we've described elements of 2 (apparently different) sets $A$ and $B$. Certainly, these descriptions are not the same, for elements of $A$ were selected by their *algebraic* properties, namely, divisibility by 2, while elements of $B$ were selected by their geometric arrangement on a line, namely, the arrangement dictated by their order (see Figure 1). However, every integer (in $A$) of the form $2 \cdot q$, $q$ an integer, is an integer in $B$, that is, $x \in A$ implies $x \in B$, or $A \subset B$. Every integer of $B$ is also of the form $2 \cdot q$ for some integer $q$, that is, $x \in B$ implies $x \in A$, or $B \subset A$. That is, $A = B$.

***Definition 6*** The **empty set** (void set, null set), symbolized by the letter $\varnothing$, is the set that has no elements.

Note that Definition 4 concerned itself with a certain relationship (inclusion) between pairs of sets. In what follows, we will define 2 operations on pairs of sets (union and intersection) that will allow us to generate a third set.

***Definition 7*** Suppose $A$ and $B$ are sets. The (third) set, ***A* union *B***, symbolized $\boldsymbol{A \cup B}$, is the set of elements that belong either to $A$ or to $B$ (or possibly to both $A$ and $B$). In other words an element qualifies as a member of $A \cup B$ if and only if that element is already an element of (*at least*) one (but not necessarily both) of the sets, $A$ or $B$. Thus,

$$x \in A \cup B \quad \text{if and only if} \quad x \in A \quad \text{or} \quad x \in B.$$

**Example 5** Let $A = \{4,a,D\}$ and $B = \{a,b,c\}$. Then $A \cup B = \{4,a,D,b,c\}$. Note that $a \in A$ and $a \in B$, but it is not "counted twice" in $A \cup B$. The element $a$ meets the test of Definition 7 in that it is an element of at least one of the sets, $A$ or $B$, and that's all we need to know.

**Example 6** Let $A = \{1,2,3,4\}$, and $B = \{a,b,c\}$. Then $A \cup B = \{1,2,3,4,a,b,c\}$.

***Definition 8*** Suppose $A$ and $B$ are sets. The (third) set, ***A* intersect *B***, symbolized $\boldsymbol{A \cap B}$, is the set of elements that belong to both $A$ and $B$ (at the same time). Thus,

$$x \in A \cap B \quad \text{if and only if} \quad x \in A \quad \text{and} \quad x \in B.$$

**Example 7** Let $A$ and $B$ be as in Example 5. Looking for the elements which are common to both sets, we are able to unearth only the element $a$. Thus, $A \cap B = \{a\}$.

**Example 8** Let $A$ and $B$ be as in Example 6. There are absolutely no elements common to *both* $A$ and $B$ in this case. Thus $A \cap B = \varnothing$, the set with no elements (Definition 6).

Sets of the form {3,chair, #} are not frequent, and, indeed, may raise the question, "Is any and every object eligible to be a member of a set?" Were we to answer in the affirmative, we would soon arrive at some unhappy paradoxes. For example, admitting "too many" sets leads to such puzzles as deciding whether the following sentence is true or false:

"This sentence is false."

If the sentence is true, then we are obliged to believe its statement, namely, it is a false sentence. If, on the other hand, we decide to judge the sentence as false, then we must believe the opposite of its message, namely, that the sentence is not false. Such a dilemma might impel one to abandon mathematics altogether, but a judicious limiting of eligible sets does put our house in order. It is for this reason that we prescribe all of our "eligible" elements in advance. The set of elements under consideration will be called the **universal set** (universe) and is denoted $U$. For example, $U$ may stand for all integers. We then are permitted to speak only of subsets whose elements are integers.

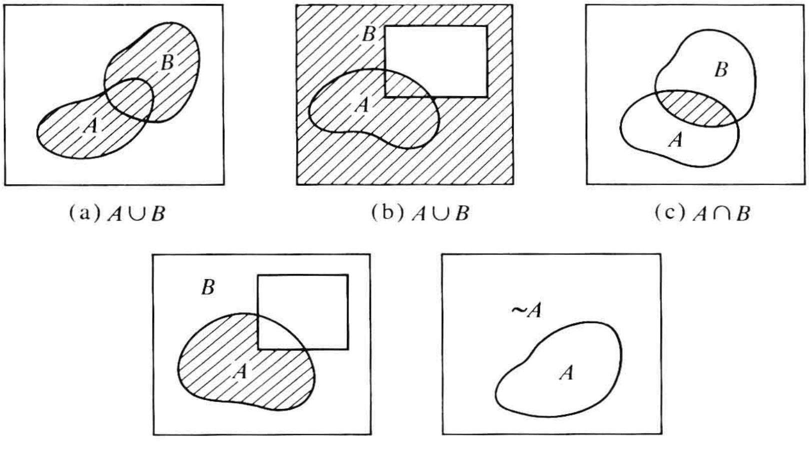
**Example 9** Let $U = \{-2, -1, 0, 1, 2\}$. In *this* context, now, the set $\{1,5\}$ has no meaning, since the element 5 is not drawn from our universe $U$.

***Definition 9*** Suppose $A$ is a set. The set **$A$ complement** or **complement of $A$** (symbolized $\sim A$), is exactly the set of elements which are in $U$, but not in $A$. Thus, $x \in \sim A$ if and only if $x \in U$ and $x \notin A$.

**Example 10** Let $U = \{s,u,v,w,3,4\}$, and let $A = \{u,v,3\}$. Then $\sim A = \{s,w,4\}$, the set of elements of $U$, which are not in $A$.

The so-called "Venn diagram" is a useful device for illustrating the previously defined operations and set relations. The idea of a Venn diagram is to present a set by a certain set of points in the plane (a 2-dimensional "surface"). Observations (however obvious) from Venn diagrams do not *prove* anything about general sets. For one thing, not every set is, in fact, a subset of the plane, and secondly, all proofs must flow from the explicit use of the *definitions*.

We present examples of Venn diagrams in Figure 2. The shaded regions represent the sets indicated under the set $U$. Notice that in (b) and (d) the set $B$ is the set of points (in $U$) which are outside the rectangle.



(a) $A \cup B$ (b) $A \cup B$ (c) $A \cap B$ (d) $A \cap B$ (e)

***Figure 2***

## EXERCISES

Verify the following statements (Exercises 3 through 13) for sets $A$, $B$, and $C$ in the universe $U$. Use the definitions and illustrate with a Venn diagram.

**3** $A \subset A$

**4** $A \cup A = A$

**5** $A \cap A = A$

**6** $A \cap \varnothing = \varnothing$

**7** $A \cup \varnothing = \varnothing$

**8** $\varnothing \subset A$

**9** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**10** $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$

**11** $\sim(\sim A) = A$

**12** $\sim \varnothing = U$

**13** $\sim U = \varnothing$

**14** Let $U = \{1,2,3,4,5,6\}$, $A = \{1,3,5\}$, $B = \{2\}$, and $C = \{1,2,3\}$. Describe the following sets:

(a) $\sim A$, $\sim B$, $\sim C$.

(b) $A \cap B$, $A \cap \sim B$, $\sim(\sim A \cup B)$

(c) $A \cap C$, $A \cap \sim C$, $\sim(\sim A \cup C)$

(d) $A \cup B$, $A \cup \sim B$, $\sim(A \cap B)$

(e) $A \cup \sim B$, $\sim(\sim A \cap B)$

(f) $(A \cap B) \cap C$

(g) $A \cap (B \cap C)$

(h) $(A \cap \sim B) \cap \sim C$

(i) $A \cap (B \cup C)$

(j) $(A \cap B) \cup (A \cap C)$

**15** Show by example, that for certain sets $A$ and $B$, it is not always true that

$$\sim(A \cup B) = \sim A \cup \sim B.$$

Show that, however,

$$\sim(A \cup B) = \sim A \cap \sim B \quad \textit{is always true.}$$

**16** Show by example, that for certain sets $A$ and $B$, it is not always true that

$$\sim(A \cap B) = \sim A \cap \sim B.$$

Show that, however,

$$\sim(A \cap B) = \sim A \cup \sim B \quad \textit{is always true.}$$

**17** Prove that, for any sets $A$ and $B$,

$$(A \cap B) \subset B \quad \text{and} \quad (A \cap B) \subset A$$

**18** Prove that, for any sets $A$ and $B$,

$$A \subset (A \cup B) \quad \text{and} \quad B \subset (A \cup B).$$

CHAPTER I

# Vector Spaces and Applications

## 1. Addition and Scalar Multiplication of Vectors

*Definition I.1.1* A **concrete column vector** is an ordered set of $n$ numbers (or **scalars**)

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

written in vertical fashion, where $n$ is some positive integer. A **concrete row vector** is an ordered set of numbers $(x_1,x_2,\ldots,x_n)$ written in horizontal fashion. In both cases, the numbers $x_1, x_2,\ldots, x_n$, are called **coordinates, entries**, or **components**, of the respective vectors.

The set of all column vectors is denoted by the symbol $V_n$. $V_n$ is also denoted by $V_n(\mathbf{R})$ or $V_n(\mathbf{C})$ according to whether the entries are elements of the real numbers $\mathbf{R}$, or complex numbers $\mathbf{C}$, respectively.

**Example I.1**

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix}, (1,3,0,2), (1,1), (0,\tfrac{1}{2},-\tfrac{1}{2}).$$

The first 3 vectors are column vectors (in $V_3$) with 3, 2, and 4 components (coordinates), respectively. The last 3 vectors are row vectors with 4, 2, and 3 components, respectively.

**Example I.2** For each family on the block, we may construct a column (or row) vector with 3 coordinates. Let the first coordinate represent the pennies spent each week on milk, the second coordinate represent the weekly expenditure for garlic, and the third coordinate agree with the weekly expenditure for mouthwash. Thus, if there are only 2 families on the block, family $A$ and family $B$, we might represent their weekly expenditures:

$$X_A = \begin{pmatrix} 250 \\ 10 \\ 60 \end{pmatrix} \qquad X_B = \begin{pmatrix} 125 \\ 40 \\ 730 \end{pmatrix} \begin{matrix} \leftarrow \text{milk} \\ \leftarrow \text{garlic} \\ \leftarrow \text{mouthwash.} \end{matrix}$$

Suppose we raise the question: How much money is spent by the families on the block in a week? It seems clear that to find the *combined* weekly expenditure profile, we add componentwise, that is, if expenses of family $A = X_A$, expenses of family $B = X_B$, then

$$X_A \text{ plus } X_B = \begin{pmatrix} 250 \\ 10 \\ 60 \end{pmatrix} \text{ plus } \begin{pmatrix} 125 \\ 40 \\ 730 \end{pmatrix} = \begin{pmatrix} 250 + 125 \\ 10 + 40 \\ 60 + 730 \end{pmatrix} = \begin{pmatrix} 375 \\ 50 \\ 790 \end{pmatrix} \begin{matrix} \leftarrow \text{milk} \\ \leftarrow \text{garlic} \\ \leftarrow \text{mouthwash.} \end{matrix} \qquad \text{(I.1.1)}$$

Actually, Equation (I.1.1) sets the pattern for the notion of *addition of one vector to another*. We set down a formal definition of vector addition with

***Definition I.1.2*** Suppose $X$ and $Y$ are column vectors, each with $n$ coordinates. Then the **sum of X and Y, X ∔ Y**, is defined by

$$X \dotplus Y = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \dotplus \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n + y_n \end{pmatrix}$$

Similarly, if $X$ and $Y$ are each row vectors with $n$ coordinates, then **X plus Y, X ∔ Y**, is defined by

$$(x_1,x_2,\ldots,x_n) \dotplus (y_1,y_2,\ldots,y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

REMARK. The sum of a pair of concrete vectors is meaningful only when both vectors have the *same* number of components (coordinates). Moreover, we observe that the sum of 2 vectors, each having $n$ components, yields a vector of the *same type*, that is, one with $n$ components, Also, we have used the symbol ∔ to underscore the fact that we are adding something *other* than numbers; addition of numbers is denoted by the usual symbol +. The following example will motivate our definition of what is meant by the multiplication of a vector by a number.

**Example I.3** Suppose we construct a "happiness profile" for a certain child over a year's time which lists the following: (1) The number of pounds of popcorn received, and (2) the number of hours spent juggling oranges. If, in the year 1878, our child received 21 pounds of popcorn, and spent 48 hours juggling oranges, then his happiness profile would be represented by the column vector

$$X = \begin{pmatrix} 21 \\ 48 \end{pmatrix} \begin{matrix} \leftarrow \text{pounds of popcorn received during 1878} \\ \leftarrow \text{hours juggling oranges in 1878.} \end{matrix}$$

Now if, in the following year of 1879, we desired to "increase the happiness" of the child $2\frac{1}{2}$ times, we could reasonably expect to achieve this goal by multiplying each component of the happiness profile vector, $X$, by the number $2\frac{1}{2}$. That is,

$$2\tfrac{1}{2} \text{ times} \begin{pmatrix} 21 \\ 48 \end{pmatrix} = \begin{pmatrix} 2\frac{1}{2} \cdot 21 \\ 2\frac{1}{2} \cdot 48 \end{pmatrix} = \begin{pmatrix} 52\frac{1}{2} \\ 120 \end{pmatrix},$$

which tells us that in the year 1879, this child will have been $2\frac{1}{2}$ times as happy as he was the previous year, after receiving a total of $52\frac{1}{2}$ pounds of popcorn, and after