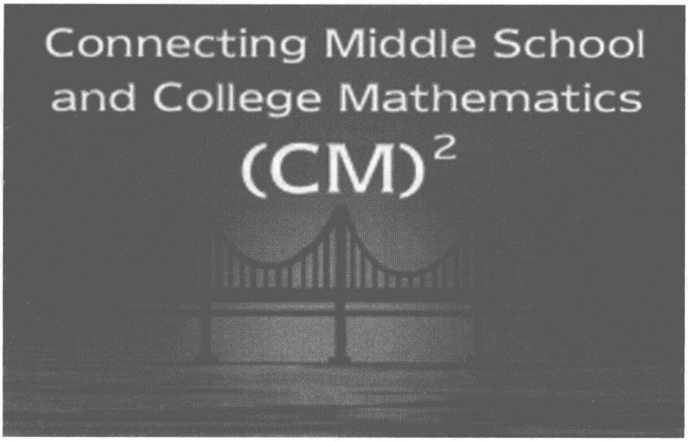
ALGEBRA CONNECTIONS

MATHEMATICS FOR MIDDLE SCHOOL TEACHERS

IRA J. PAPICK

CONNECTIONS IN MATHEMATICS COURSES FOR TEACHERS



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Mathematics for Middle School Teachers

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Algebra Connections

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Preface

Improving the quality of mathematics education for middle school students is of critical importance, and increasing opportunities for students to learn important mathematics under the leadership of well-prepared and dedicated teachers is essential. New standards-based curriculum and instruction models, coupled with on-going professional development and teacher preparation, are foundational to this change.

These sentiments are eloquently articulated in the Glenn Commission Report: *Before It's Too Late: A Report to the Nation from the National Commission on Mathematics and Science Teaching for the 21st Century* (U.S. Department of Education, 2000). In fact, the principal message of the Glenn Commission Report is that America's students must improve their mathematics and science performance if they are to be successful in our rapidly changing technological world. To this end, the Report recommends that we greatly intensify our focus on improving the quality of mathematics and science teaching in grades K–12 by bettering the quality of teacher preparation, and it also stresses the necessity of developing creative plans to attract and retain substantial numbers of future mathematics and science teachers.

Some fifteen years ago, mathematics teachers, mathematics educators, and mathematicians collaborated to develop the architecture for standards-based reform, and their recommendations for the improvement of school mathematics, instruction, and assessment were articulated in three seminal documents published by the National Council of Teachers of Mathematics (*Curriculum and Evaluation Standards for School Mathematics* [1989], *Professional Standards for School Mathematics* [1991], and *Assessment Standards in School Mathematics* [1995]; more recently, these three documents were updated and combined into the single book, *NCTM Principles and Standards for School Mathematics, a.k.a. PSSM* [2000]).

The vision of school mathematics laid out in these three foundational documents was outstanding in spirit and content, yet abstract in practice. Concrete exemplary models reflecting the standards were needed and implementing the recommendations would be unrealizable without significant commitment of resources. Recognizing the opportunity for stimulating improvement in student learning, the National Science Foundation (NSF) made a strong commitment to bring life to the documents' messages and supported several K–12 mathematics curriculum development projects (standards-based curriculum), as well as other related dissemination and implementation projects.

Standards-based middle school curricula are designed to engage students in a variety of mathematical experiences, including thoughtfully planned explorations that provide and reinforce fundamental skills while illuminating the power and utility of mathematics in our world. These materials integrate central concepts in algebra, geometry, data analysis and probability, and mathematics of change, and they focus on important unifying ideas such as proportional reasoning.

The mathematical content of standards-based middle grade mathematics materials is challenging and relevant to our technological world. Its effective classroom implementation is dependent upon teachers having strong and appropriate mathematical preparation. *The Connecting Middle School and College Mathematics Project* $(CM)^2$ is a three-year (2001–2004) National Science Foundation funded project addressing the need for improved teacher qualifications and viable recruitment plans for middle grade mathematics teachers through the development of four foundational mathematics courses with accompanying support materials and the creation and implementation of effective teacher recruitment models.

The $(CM)^2$ materials are built upon a framework laid out in the *CBMS Mathematical Education of Teachers Report* (MET) (2001). This report outlines recommendations for the mathematical preparation of middle grade teachers that differ significantly from those for the preparation of elementary teachers and provides guidance to those developing new programs. Our books are designed to provide middle grade mathematics teachers with a strong mathematical foundation and connect the mathematics they are learning with the mathematics they will be teaching. Their focus is on algebraic and geometric structures, data analysis and probability, and mathematics of change, and they employ standards-based middle grade mathematics curricular materials as a springboard to explore and learn mathematics in more depth. They have been extensively piloted in Summer Institutes, in courses offered at school-based sites, through a variety of professional development programs, and in both undergraduate and graduate semester courses offered at a number of universities throughout the nation.

This book is written as an introduction to some basic concepts of number theory and modern algebra that underlie middle grade arithmetic and algebra, and thus the approach differs from some traditional texts in these subjects. The primary goal is to help teachers (both in-service and pre-service) gain a fundamental understanding of the key mathematical ideas that they will be teaching, so that in turn they can help their students learn important mathematics.

Throughout the book, the reader will find a number of **Classroom Connections, Classroom Discussions**, and **Classroom Problems**. These instructional components are designed to deepen the connections between the algebra and number theory students are studying now and the algebra they will teach. The **Classroom Connections** are middle grade investigations that serve as launch pads to the college level **Classroom Discussions, Classroom Problems,** and other related collegiate mathematics. The **Classroom Discussions** are intended to be detailed mathematical conversations between college teacher and pre-service middle grade teachers, and are used to introduce and explore a variety of important concepts during class periods. The **Classroom Problems** are a collection of problems with complete or partially complete solutions and are meant to illustrate and engage pre-service teachers in various problem solving techniques and strategies. The continual process of connecting what they are learning in the college classroom to what they will be teaching in their own classroom provides teachers with real motivation to strengthen their mathematical content knowledge.

Many of my recent students studied from preliminary versions of these materials, and their thoughtful comments significantly shaped the contents of this

book. I am most grateful to these present and future teachers and take great pride in their mathematical growth. I am also thankful for the insightful suggestions of the many mathematicians and mathematics educators who piloted these materials in their college classrooms or in professional development venues. I am especially appreciative to Professors Jennifer Bay-Williams, Kansas State University; Al Dixon, Western Michigan University; and Steve Ziebarth, College of the Ozarks, for their careful reviews of a preliminary version of this text. Their astute and detailed remarks notably improved the materials. Writing this book has been a great joy. The mathematical adventure was especially exciting and having the opportunity to work with outstanding graduate students was an incredible bonus. I am deeply thankful to David Barker for crafting a comprehensive first draft of Chapter 1. He and I spent countless hours discussing the learning and teaching of mathematics, and we learned a great deal from each other. I would also like to extend my sincere gratitude to graduate students Dustin Foster and Chris Thornhill, to middle grade teacher Paul Rahmoeller, and to post-doctoral fellow, Jason Aubrey for reading (re-reading, re-re-reading,. . .) over the manuscript, solving selected exercises, and making many valuable suggestions. Finally, I am most appreciative to Petra Recter at Pearson/Prentice-Hall for her expert assistance in bringing this book to print.

Ira J. Papick

Contents

Patterns

CHAPTER 1

In a broad sense, the study of patterns and relationships is the essence of mathematics and, accordingly, it occupies a central position in school mathematics. Mathematicians seek to understand fundamental structures by searching for patterns and relationships within classes of examples and collections of data. Their investigations involve insightful questions and conjectures in unison with creative thinking and problem-solving strategies, and it is especially crucial for all students of mathematics to comprehend and embrace these habits of discovery.

1.1 CLASSROOM CONNECTIONS: REPRESENTING PATTERNS

We begin this chapter by looking at the Tiling Pools problem from the eighth grade module *Say It with Symbols* of the *Connected Mathematics* curriculum. As you work through this problem (as well as through other middle-school problems throughout this textbook), pay special attention to the following questions:

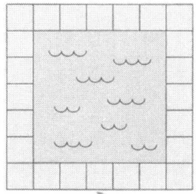
1. What strategies did you use to solve the problem, and what strategies do you think students will use?

2. What types of rules did you discover, and what types of rules do you think students will produce?
3. How did you justify your rules, and what types of justifications do you expect your students to give?
4. What counts as an acceptable justification at the middle-school level?

This problem and others provide the basis of many of our discussions throughout this chapter and illuminate many important ideas concerning patterns.

2.1 Tiling Pools

Hot tubs and in-ground swimming pools are sometimes surrounded by borders of tiles. This drawing shows a square hot tub with sides of length 5 feet surrounded by square border tiles. The border tiles measure 1 foot on each side. A total of 24 tiles are needed for the border.



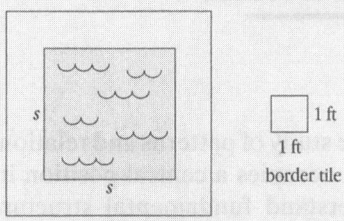
20 **Say It with Symbols**

Reproduced from page 20 of *Say It with Symbols* in *Connected Mathematics.*

FIGURE 1.1.1

Problem 2.1

In this problem, you will explore this question: If a square pool has sides of length *s* feet, how many tiles are needed to form the border?



A. Make sketches on grid paper to help you figure out how many tiles are needed for the borders of square pools with sides of length 1, 2, 3, 4, 6, and 10 feet. Record your results in a table.

B. Write an equation for the number of tiles, *N*, needed to form a border for a square pool with sides of length *s* feet.

C. Try to write at least one more equation for the number of tiles needed for the border of the pool. How could you convince someone that your expressions for the number of tiles are equivalent?

Reproduced from page 21 of *Say It with Symbols* in *Connected Mathematics.*

FIGURE 1.1.2

Problem 2.1 Follow-Up

1. Make a table and a graph for each equation you wrote in part a of Problem 2.1. Do the table and the graph indicate that the equations are equivalent? Explain.
2. Is the relationship between the side length of the pool and the number of tiles linear, quadratic, exponential, or none of these? Explain your reasoning.
3. **a.** Write an equation for the area of the pool, A, in terms of the side length, s.
   b. Is the equation you wrote linear, quadratic, exponential, or none of these? Explain.
4. **a.** Write an equation for the combined area of the pool and its border, C, in terms of the side length, s.
   b. Is the equation you wrote linear, quadratic, exponential, or none of these? Explain.

Reproduced from page 21 of *Say It with Symbols* in *Connected Mathematics*.

FIGURE 1.1.3

1.2 REFLECTIONS ON CLASSROOM CONNECTIONS: REPRESENTING PATTERNS

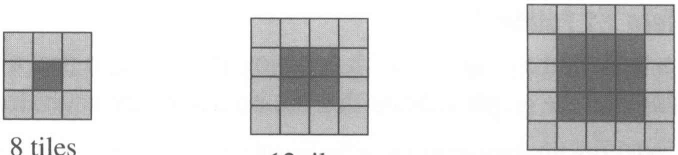
It is common for students to think about and solve mathematics problems in a multitude of ways. For example, here are the thoughts of eighth graders Meaghan and Reese on the problem of determining the number of square tiles (1 foot by 1 foot) needed to form the boundary of a square pool of dimensions s feet by s feet (where s is a positive integer).

Meaghan. I drew out the first three pools and noticed that each time you add a foot to the side of the pool, the number of tiles goes up by 4.

Reese. I noticed that for a pool of any size you will always have a tile for each foot of the perimeter, or $4n$, and then you need 4 more tiles for the corners, so I added 4.

Meaghan and Reese have taken different approaches in solving this problem, and it is instructive to look at their responses in more detail.

Meaghan's Strategy. Meaghan initially drew pictures of square pools with side lengths 1, 2, and 3 feet. Once she drew these examples and calculated the number of tiles needed to surround the pools, she compared the results and conjectured a relationship between pools with consecutive integer side lengths.



By looking at the pattern 8 tiles, 12 tiles, and 16 tiles, she concluded that, "each time you add a foot to the side of the pool, the number of tiles goes up by 4." Hence, if you knew the number of tiles required to surround a square pool of length 9 feet (which can be determined from the previous cases), you could then find the number of tiles required to surround a square pool of length 10 feet by simply adding 4. Moreover, since you know the number of tiles for a square pool of length 1 foot, you can determine the number of tiles for all whole number length square pools. Why?

Classroom Problem. Using Meaghan's rule, determine how many tiles are needed for a square pool of length 9 feet. Represent your data in a table format (as here).

Side Length in Feet	Number of Tiles
1	8
2	12
3	16
4	
5	
6	
7	
8	
9	

The primary advantage of Meaghan's rule is that it is easy to calculate the number of tiles for a square pool of length n provided you know the number of tiles for a square pool of length $n - 1$, while the main disadvantage is that it is difficult to determine the number of tiles for larger length square pools (e.g., for a square pool of length 2,467 feet). ◆

Question. Meaghan arrived at her rule by inspecting some particular examples and did not show that her rule holds for all positive integer lengths. How would you justify the validity of Meaghan's rule for all positive integer lengths?

The kind of pattern that Meaghan observed—where after some explicit terms are specified, each subsequent term is defined in terms of a previous term or a combination of previous terms—is called a **recursive pattern**. The rule that describes the relationship between these consecutive terms is called a **recursive rule or formula**.

Representing Rules. The middle grades are an important time for students as they begin to develop the ideas of variable and function. The gradual transformation from describing rules using language to describing rules using symbols is a key transition during this time. Notation, which is intended to simplify thinking, can often be confusing to students during their initial exposure because they perceive the notion of *variable* in a variety of ways. Hence, an appropriate understanding of what these representations mean and how they are used is a must for students.

Although Meaghan described her rule in words, it is possible to express it in symbols. This kind of representation is especially useful for more complicated rules, since it compresses information into notation that is more workable. For example, if we let T_1 represent the number of tiles in a square pool of length 1 foot (the value of the pattern's first term), T_2 represent the number of tiles in a square pool of length 2 feet (the value of the pattern's second term), etc., then Meaghan's recursively defined rule for the pool problem could be stated as follows:

$$T_1 = 8$$
$$T_n = T_{n-1} + 4, \ (n > 1)$$

For this rule, T_n represents the number of tiles needed to surround a square pool with a side of length n (the value of the n^{th} term of the pattern), and T_{n-1} is the number of tiles required for a square pool of length $n - 1$ (the value of the $(n - 1)^{\text{th}}$ term of the pattern).

Classroom Problem. Let's write a recursive rule for the pattern that occurs in the following problem.

Farmer Jim (or Jimbo as he is called by his closest friends) uses fence panels of the same length to create pens for his animals. He decides to arrange the pens in a single row with all the pens being connected as illustrated in the picture here.

The number of fence panels needed for these three pens is recorded in the following table.

Term (number of animal pens)	**Value** (number of panels required)
1	4
2	7
3	10

We see that it takes four panels to create the first animal pen, and this can be expressed in notation as $P_1 = 4$. Next, we need to find a relationship between consecutive terms of this pattern. If an additional pen is appended to the first pen, Farmer Jim will use one panel from the end of the first pen and add three more panels to get to the required four panels to complete the second pen. Similarly, three more panels are needed to create the third pen, and so the n^{th} pen is built by adding three panels to the $(n - 1)^{\text{th}}$ pen. This relationship can be expressed as

$$P_n = P_{n-1} + 3,$$

and so the complete recursive formula describing this situation is given by:

$$P_1 = 4$$
$$P_n = P_{n-1} + 3. \quad \blacklozenge$$

Reese's Strategy. Recall that Reese's approach to the Tiling Pools problem differed from Meaghan's strategy. He states, "For a pool of any size, you will always have a tile for each foot of the perimeter, or $4n$, and then you need four more tiles for the corners." Instead of comparing the number of tiles needed for a few different-length square pools (as Meaghan did), Reese developed a systematic way of counting the tiles needed for each square pool of length n (n a positive integer). The rule Reese developed establishes an explicit relationship between the length of a side of the pool and the number of tiles required to surround it.

n = positive integer length (in feet) of a square pool	$4n + 4$ = number of tiles in the boundary of the pool
1	$4 \bullet 1 + 4 = 8$
2	$4 \bullet 2 + 4 = 12$
3	$4 \bullet 3 + 4 = 16$
4	$4 \bullet 4 + 4 = 20$
5	$4 \bullet 5 + 4 = 24$

In mathematical terms, Reese's **explicit rule** defines a function T on the set of positive integers, given by $T(n) = 4n + 4$, where n is a positive integer length (in feet) of a square pool, and $T(n)$ is the total number of tiles needed for a square pool of length n.

In general, a function f whose domain is the positive integers (into any other set) is called an **infinite sequence** (or simply a **sequence**). The **range of a sequence**,

$$\textbf{Range of } f = \{f(n): n \text{ is a positive integer}\},$$

is usually written in the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

where $f(n) = a_n$ for each positive integer n. The sequence that Reese discovered is

$$8, 12, 16, 20, 24, \ldots, 4n + 4, \ldots,$$

where n is a positive integer.

Convention. Since a sequence's domain is always the positive integers, it is common practice to identify a sequence f with its range: $a_1, a_2, a_3, \ldots, a_n, \ldots$

Using Reese's explicit rule, it is easy to calculate the number of tiles needed for a square pool of length n. This is a benefit of his rule over Meaghan's recursive rule. However, as we shall see when we study the Fibonacci sequence (Section 1.11), it is not always straightforward to determine an explicit rule for a given sequence.

An Explicit Rule for Farmer Jim. Let's return to the problem of Farmer Jim's livestock pens, but this time we try to describe the number of pens with an explicit rule rather than a recursive rule.

The following table consists of some conclusions we have drawn from looking at specific cases, which may be useful in formulating a specific rule.

Number of Pens	Panels Required
1	4
2	7
3	10
4	13
5	16

Looking at the table of values, it might be conjectured that the number of panels required for any (positive integer) number of pens n is $P(n) = 3n + 1$; however, we cannot draw this conclusion based solely on these few cases.

It is common for (middle grade) students to create rules based upon only a few cases, often a single case. They might look at the previous table and see that 3 pens require 10 panels and thus state that the general rule is $P(n) = n^2 + 1$, which as it turns out, only happens to work for this particular case. It is important for all students of mathematics to understand that they must justify general conclusions through valid arguments and not rely exclusively on the verification of a few cases. Since we cannot guarantee from the table's information that the explicit rule describing the pattern of this problem is $P(n) = 3n + 1$, let's turn to the context of the problem to assist us in justifying this rule.

Justification. One way to construct n pens is to put together n groups of three-sided pens and join them in the manner illustrated here (for $n = 4$). This grouping requires $3n$ panels and lacks one panel to close off the last pen. Hence, the expression $3n + 1$ gives the total number of panels required for n pens.