Sets: Naive, Axiomatic and Applied

A Basic Compendium with Exercises for use in Set Theory for Non-Logicians, Working and Teaching Mathematicians and Students.

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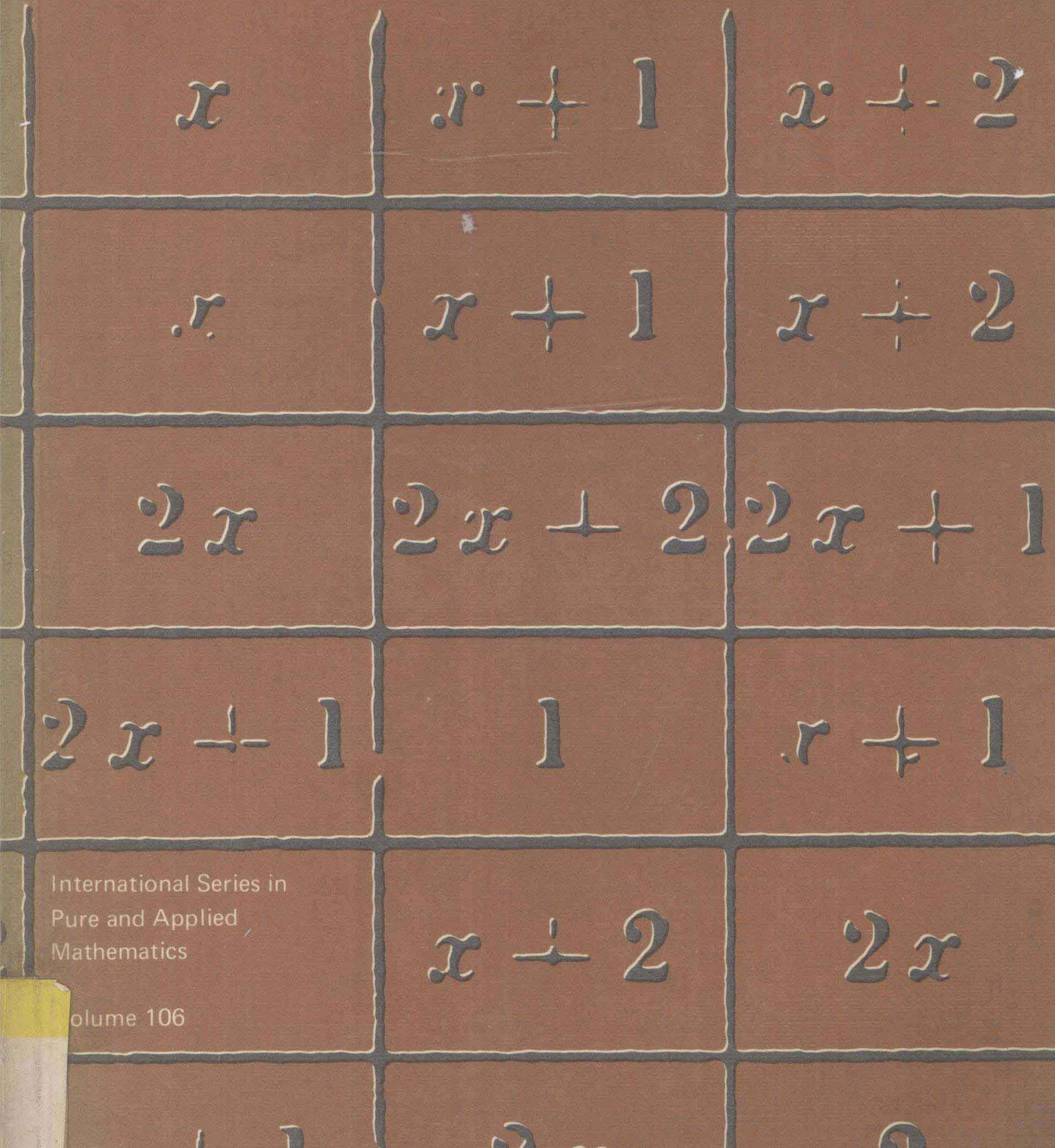

International Series in Pure and Applied Mathematics

Volume 106

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PERGAMON PRESS
OXFORD · NEW YORK · TORONTO · SYDNEY · PARIS · FRANKFURT

U.K.	Pergamon Press Ltd., Headington Hill Hall, Oxford OX3 0BW, England
U.S.A.	Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523, U.S.A.
CANADA	Pergamon of Canada Ltd., 75 The East Mall, Toronto, Ontario, Canada
AUSTRALIA	Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street, Rushcutters Bay, N.S.W. 2011, Australia
FRANCE	Pergamon Press SARL, 24 rue des Ecoles, 75240 Paris, Cedex 05, France
FEDERAL REPUBLIC OF GERMANY	Pergamon Press GmbH, 6242 Kronberg-Taunus, Pferdstrasse 1, Federal Republic of Germany

First edition 1978

British Library Cataloguing in Publication Data

Van Dalen, D
Sets: naive, axiomatic, and applied.
(International series in pure and applied mathematics; v. 106)
Translation of Verzamelingen.
Bibliography: p.
1. Set theory. 2. Axiomatic set theory.
I. Doets, H. C., joint author. II. Swart,
H. C. M. de, joint author. III. Title.
QA248.D23413 511'.32 76-43992

ISBN 0-08-021166-6 (hardcover)
ISBN 0-08-023047-4 (flexicover)

In order to make this volume available as economically and as rapidly as possible the author's typescript has been reproduced in its original form. This method unfortunately has its typographical limitations but it is hoped that they in no way distract the reader.

Translated from the Dutch edition entitled "VERZAMELINGEN" published by OOSTHOEK, SCHELTEMA & HOLKEMA (Utrecht)

Printed in Great Britain by William Clowes & Sons Limited, London, Beccles and Colchester

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INTERNATIONAL SERIES IN
PURE AND APPLIED MATHEMATICS
General Editor I. N. SNEDDON

VOLUME 106

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PREFACE

Set theory is a funny discipline. For ages and ages mathematics has managed without set theory, but nowadays one gets from the average textbook the impression that set theory is absolutely indispensable. Even texts for high schools (not to mention nursery schools) start with sets, unions, intersections, etc.

Among the professional mathematicians there are some who claim that "there are no things but sets " How do these extremists justify their opinion? We will try to unearth some of the motivation for a belief in the supremacy of set theory.

The original concept of a set was very liberal: a set could contain both frogs and functions. In mathematical language, why shouldn't a set contain points, numbers, and functions at the same time? A set was obtained by simply throwing together a number of things, or by giving some common characterization. Cantor presented the following definition of a set: A set is a collection of certain distinct objects of our intuition or of our thought into a whole [Cantor, 1895, 1966]. Clearly, therefore, one should first have objects to be able to form sets. Now then, mathematics provided a wealth of objects: numbers, points, functions, matrices, curves, etc.; so there seemed not to be any special reason to worry about the universe of set theory (apart from some disagreement concerning notions of infinity). The obvious role of set theory was one of utility and hygiene. Set theory enabled one to give clean and concise formulations. In short, the availability of the set theoretic apparatus proved to be a methodological blessing. In particular, set theory turned out to be indispensable in topology. At the same time the subject acquired, under the hands of Georg Cantor, an interest of its own. In his relatively short but extremely fruitful period of creativity Cantor introduced almost all the standard concepts of our present-day set theory: cardinal, ordinal, well-ordering, powerset, etc. (for an historical survey see Van Dalen-Monna, 1972). All the same, for the time being set theory depended on a number of expedients from various parts of mathematics such as "natural number" and "function". The concept of a function, in particular, was very prominent, if only because of its role in the concepts of "cardinal", "ordertype", etc. In the early stage, therefore, set theory was certainly not a suitable base for mathematics. Set theory itself was in need of extraneous elements.

The function concept already had a long history. In the nineteenth century it had evolved from "analytical expression" to Dirichlet's famous abstract version (cf. Monna, 1972), but there was no reduction to the concept of set. Terms like "correspondence" and "Zuordnung" were used without analyzing them. The definition of a function, as a set of ordered pairs became available only after the discovery that "ordered pair" could be defined intrinsically in terms of sets (Wiener, Hausdorff, Kuratowski).
Moreover, through practical experience mathematicians found that various well-known concepts could be defined in set theoretical terms (in particular the fundamental number systems). Gradually one got the impression that the whole of mathematics could be reduced to set theory, a kind of experimental hypothesis! In due course one could observe a singular phenomenon: genuine mathematical objects were replaced by their set theoretical codifications. The success of this program can be inferred from the fact that many mathematicians take the codification for the original object. The case of the function is paradigmatic: the starting point - viz. the law (or instruction) associating objects to objects - has a clear intuitive content. Through experience we learnt that in almost all cases the manipulations of functions were independent of the defining laws, that is to say, it was sufficient to know all pairs (input, output) ! From then on the course of things was reversed: the correspondence - concept was suppressed and a function was nothing but a set of ordered pairs (a graph). One can put a label on this phenomenon: replacement of *intension* by *extension*.

This process of extensionalizing has (correctly) been questioned since the rise of computerscience. From some view-points (especially the practical one) a law (program) is more basic than a graph. In pure mathematics the intensional aspects of the function concept have been traditionally exploited by recursion theory.

Let us try to summarize the arguments for the slogan "Everything is a set": all objects from mathematical practice turn out to be representable in terms of sets and all current mathematical reasoning can be formalized in set theory. The above statement, which is not a mathematical theorem but rather a hypothesis concerning mathematics, is rather vague. In order to make the statement (at least in principle) provable or disprovable, we would have to formalize "mathematics" and "set theory", so that subsequently an interpretation of the first in the latter could be given. There are however fundamental and technical obstacles. A formalization of the whole of mathematics is a chimera, since we cannot, once and for all, decide the extent of mathematics; and also because no formalization can produce all true statements (by Gödel's theorem).

Therefore we will not worry about the status of the above slogan. It suffices that the set theoretical apparatus is practical and elegant. We will take set theory to be "open ended", in the sense that, if necessary, new principles can be added.

In mathematics, several axiom systems for set theory are in use. We will stick to the set theoretical universe cosist of?

We have already argued that the familiar mathematical objects can be reduced to sets. This leaves the following question unanswered: which objects (i.e. sets) should we have in order to get everything we want? In other words: what does the set theoretical universe consist of?

The construction of the set theoretical universe happens to be an adventure fit for a baron Von Münchhausen. Even leaving the infinity axiom aside (i.e. without committing ourselves to the existence of infinite objects), we can, starting from the empty set, create by means of the usual set theoretical operations a surprisingly rich fauna of objects (the hereditarily finite sets, cf. II. 11.23). The cummulative hierarchy shows how far one gets with how little. **ZF** gives you a lot of mileage.

Recent developments in set theory, in particular those of Cohen and Gödel concerning the axiom of choice and the continuum hypothesis, made it clear that Cantor's Paradise is not the comfortable abode it was advertised to be. The independence of the axiom of choice and the continuum hypothesis of respectable systems, such as those of Zermelo-Fraenkel or Von Neumann - Bernays - Gödel, makes situations possible, which are familiar in older branches of mathematics. In geometry, for instance, one has to distinguish between Desarguian and non-Desarguian geometry. So maybe, we will eventually have to adopt the practice of specifying our set theoretical axioms in more detail. Think of doing group theory in a non-Cantorian set theory. These are not purely theoretical reflections as may be illustrated by the case of Lebesgue-measure. Assuming the axiom of choice, we can prove that not every set of reals is measurable, but assuming the axiom of determinateness we can show each set of reals to be measurable. So there are consequences for daily life.

In this book we will not venture to deal with the independence problems of the "higher" axioms of set theory. We will stick to those parts that do not require a refined metamathematical apparatus. Although the authors do not subscribe to the thesis that "everything is a set" (they are not sets themselves (after A. Mostowski)), they are convinced that the fruitfulness of set theory, as a mathematical discipline, is beyond dispute. So they welcome the reader, with a clear conscience, to Cantor's paradise.

ACKNOWLEDGEMENTS

The authors wish to thank Jeff Zucker for his very helpful criticism and advice, Mrs. E.L.M. Remers for her patient and excellent typing of the manuscript and Mr. F. Ribot and J.M. Geertsen for giving their kind permission to make use of the facilities of the Faculty of Mathematics and Science of the Catholic University at Nijmegen, the Netherlands.

INTRODUCTION

The addition of another text in set theory to those already existing is an act that needs justification. The authors of the present book are confident that the contents offer a sufficient choice of topics not covered by standard textbooks in set theory, to exclude the possibility of a mere repetition of well-known stories and tricks. It is hardly necessary to stress the fundamental role of set theory in the mathematics of these days. Ever since the days of Cantor, algebra, analysis, etc., have been absorbing the techniques and the flavour of set theory, so that a journey into modern mathematics without a firm background in set theory would be ill-advised.

This book consists of three chapters: The first contains the traditional material (e.g. Boolean operations, countability, Cantor-Bernstein, well-ordering), treated informally in such a way that everything can easily be adapted to an axiomatic treatment. At the end of the first chapter the axioms of Zermelo-Fraenkel are introduced and it is indicated how to rework the preceding sections.

The second chapter is strictly axiomatic, but rather along the lines of Hilbert's *Foundations of Geometry* than along those of a formal first-order theory. In particular no deep facts of logic are used, or presupposed. A knowledge of the meaning of the connectives and some experience in mathematics suffices. Nonetheless some non-trivial subjects are treated, such as the reflection principle, measurable cardinals, and models of set theory. Also, the theory of ordinals is put on a firm base. In particular the reader may benefit from the treatment of the cumulative hierarchy (due to Von Neumann), which provides extra insight into the set-theoretical universe. Throughout, the axiom of regularity (foundation) is assumed, and its role is discussed in Section II.1.

The last chapter contains a number of topics, such as Borel sets, inductive definitions, Boolean algebras, applications of the axiom of choice, infinite games and the axiom of determinateness, which illustrate the actual use of set theory in mathematics. Many of these topics go unnoticed in an ordinary mathematics text, so the authors thought it worthwhile to include them in the book as a demonstration of the usefulness of set theory in everyday mathematics.

From the above it may be clear that the book should be useful for mathematicians, teachers and students. In addition it offers a number of topics of interest to other scientists, such as linguists and biologists. Philosophers, too, will find quite a lot of material, that has traditionally called their attention. For example, the introduction of the fundamental number systems, cardinals (Frege-Russell), the notions of finite and infinite, inductive definitions (the notion of predicativity) and the paradoxes of Cantor and Russell.

No attention is paid to deeper metamathematical topics, such as Gödel's constructible sets or Cohen's forcing. For those topics the reader is referred to the existing literature (however, cf. exercise 2 of Section II.12).

Many exercises are added and the reader is strongly advised to try his hand at them.

For the convenience of the casual reader the book contains a number of redundancies. Definitions, for example, will be repeated when the notion turns up in a different context.

In the text several terms, such as "property", "operation", "class", etc., are used. This may seem confusing at first, but, when used judiciously, it makes life much more pleasant. In the Appendix the use of these terms is explained. The reader is asked to turn to the Appendix whenever he feels uneasy in handling these terms. Rather than reading the Appendix last, he should every now and then consult it. Also, the Appendix deals with some peculiarities that stem from the linguistic limitations of the theory.

How to use the book. Those interested in a guided introduction into set theory can read Chapter I (possibly skipping Sections 12 and 21) and in addition Chapter II, Sections 9, 10. Sections of Chapter II and Chapter III can be added if desired. The reader who is already familiar with naïve set theory can read Chapter I, Section 20 and go on to Chapter II. If he wants to restrict himself to the bare minimum, he can, after reading Section 2, stick to the Sections 3, 8, 9, 10, which are of central importance. Next, he can add to this Chapter II, Sections 4, 5 and 6.

Those interested in the foundations of mathematics should not skip anything (with the possible exception of Chapter II, Section 15).

The purely mathematically oriented reader can already on the basis of Chapter I turn to the applications in Chapter III. In particular, Chapter III, Section 3 will give him some idea how to handle ordinals.

References and cross-references. We refer to the Bibliografy by a name, followed by a date, in square brackets, e.g. [Gödel, 1940]. Cross references to a section (Theorem, Lemma, Definition, etc.) in the same chapter are made without mentioning the chapter, e.g. see Section 5 (see Section 5.3). Cross references to other chapters give the number of the chapter, e.g. see Section II.5 (see Section II.5.3).

Problems. The book contains a number of exercises of various degrees of difficulty. We do not claim any originality for the exercises. Some problems are really minor theorems and since they are quite often very useful, we strongly advise the reader to pay attention to them. Although it may seem superfluous we want to remind him that without solving problems (i.e. proving theorems) he himself will probably not get beyond the stage of collecting keywords.

Proofs. Not all proofs will be given in full detail and some proofs will altogether be left to the reader. The end of a proof is indicated by the sign □. It is hardly necessary to point out that only a fraction of the present corpus of set theory is treated in this book. Spectacular results, such as the independence of the axiom of choice and the continuum hypothesis in Zermelo-Fraenkel's system, require essential use of more sophisticated logic machinery. For these and other topics the reader is referred to [Cohen, 1966], [Gödel, 1940], [Fraenkel, Bar-Hillel, and Levy, 1973]. For those interested in the historical development of set theory it is worthwhile to trace it back to its sources. For this a reader should consult [Cantor, 1966], [Van Dalen, and Monna, 1972].

CONTENTS

Chapter III. Applications

CHAPTER 1

NAÏVE SET THEORY

1 SOME IMPORTANT SETS AND NOTATIONS

We all know lots of sets, for example: the set of all people in Holland, the set of all triangles in a plane, the set of the numbers 1, 2, 3, 4 and 5. The latter is usually denoted by $\{1, 2, 3, 4, 5\}$.

Instead of "3 is an element of the set $\{1, 2, 3, 4, 5\}$" we write "$3 \in \{1, 2, 3, 4, 5\}$"; and instead of "7 is not an element of the set $\{1, 2, 3, 4, 5\}$" we write "$7 \notin \{1, 2, 3, 4, 5\}$".

The numbers 0, 1, 2, 3,... are called *natural numbers*. By $\mathbb{N}$ we mean the set of all natural numbers. So, for example, $3 \in \mathbb{N}$, $5 \in \mathbb{N}$ and $1024 \in \mathbb{N}$, while $-3 \notin \mathbb{N}$, $2/3 \notin \mathbb{N}$ and $\sqrt{2} \notin \mathbb{N}$.

The numbers ... −3, −2, −1, 0, 1, 2, 3, ... are called *integers*. By $\mathbb{Z}$ we mean the set of all integers. Note that each natural number is an integer, but not conversely. Examples: $2 \in \mathbb{Z}$, $-2 \in \mathbb{Z}$, $0 \in \mathbb{Z}$, $3 \in \mathbb{Z}$, $-3 \in \mathbb{Z}$, $2/3 \notin \mathbb{Z}$ and $\sqrt{2} \notin \mathbb{Z}$.

The numbers of the form p/q, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \neq 0$ (and p and q relatively prime) are called *rational numbers*. By $\mathbb{Q}$ we mean the set of all rational numbers. Examples: $1/4 \in \mathbb{Q}$, $-1/4 \in \mathbb{Q}$, $2 \in \mathbb{Q}$, $-2 \in \mathbb{Q}$, $0 \in \mathbb{Q}$, $3/5 \in \mathbb{Q}$, $-3/5 \in \mathbb{Q}$, $\sqrt{2} \notin \mathbb{Q}$, $\sqrt{5} \notin \mathbb{Q}$, $\pi \notin \mathbb{Q}$. Note that all integers and hence also all natural numbers are rational numbers.

All rational numbers and numbers such as $\sqrt{2}$, $\sqrt{5}$, $\log 2$, $\sqrt[4]{7}$, π, and so on, are called *real numbers*. By $\mathbb{R}$ we mean the set of all real numbers (a precise definition follows in Section 12).

In our examples, sets consisted of concrete and familiar objects, but once we have sets, we can form sets of sets, e.g. the set of all football teams. It is important to realize that the elements of a set may themselves be sets. Mathematics is full of examples of sets of sets. A straight line, for example,