Lectures on

EQUATIONS DEFINING SPACE CURVES

By

L. SZPIRO

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
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N. MOHAN KUMAR

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Author

L. SZPIRO
Ecole Normale Superieure
Université de Paris
45, Rue D'Ulm
75230 Paris - FRANCE

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INTRODUCTION

THESE NOTES ARE the outcome of a series of lectures I gave in the winter of 1975-'76 at the Tata Institute of Fundamental Research, Bombay. The object of the research, we - D. FERRAND, L. GRUSON, C. PESKINE and I - started in Paris was, roughly speaking to find out the equations defining a curve in projective 3-space (or in affine 3-space or of varieties of codimension two in projective n-space.) I took the opportunity given to me by the Mathematics Department of T.I.F.R, to try to put coherently the progress made by the four of us since our paper [11]. Even though we are scattered over the earth now, (RENNES, LILLE, OSLO and BOMBAY!) these notes should be considered as the result of common of all of us. I have tried in the quick description of the chapters to obey the "Redde Caesari quae sunt Caesaris."

Chapter I contains certain prerequisites like duality, depth, divisors etc. and the following two interesting facts:

i) An example of a reduced curve in $\mathbb{P}^3$ with no imbedded smooth deformation (an improvement on the counter example "6.4" in [11] which was shown to me by G. Ellingrud from Oslo who also informed me that it can be found in M. Noether [10]).

ii) A proof that every locally complete intersection curve in $\mathbb{P}^3$ can be defined by four equations.

Chapter II is my personal version of the theory of conductor for a curve. A long time ago, O. Zariski asked me what my understanding of Gorenstein's theorem was and this chapter is my answer; even though it contains no valuations and I wonder if it will be to the taste of Zariski. In it I first recall classical facts known since Kodaira, through duality. The three main points are as follows:

If X is a smooth surface, projective over a field k, C, a reduced irreducible curve on X, $\bar{X} \xrightarrow{g} X$, a finite composition of dilations, such that the proper transform $\bar{C}$ of C on $\bar{X}$ is smooth, one has:

a) the conductor $\underline{f}$ is related to dualizing sheaves by

$$\underline{f} = g_* \omega_{\bar{C}} \otimes \omega_C^{-1}$$

b) Gorenstein's theorem is a simple consequence of $H^1(\mathbb{P}^2, O_{\mathbb{P}^2}) = 0$.

c) Regularity of the adjoint system is equivalent to $H^1(\bar{X}, O_{\bar{X}}(\bar{C})) = 0$.

We conclude the chapter with a counter-example which is new in the litterature:

d) A curve C on a surface X over a field of characteristic $p \geq 5$, such that

i) $O_X(C)$ is ample.

ii) Kodaira vanishing theorem holds i.e. $H^1(X, O_X(-C)) = 0$.

iii) Regularity of the adjoint does not hold.

i.e. $H^1(\bar{X}, O_{\bar{X}}(-\bar{C})) \neq 0$.

We also give the proof - shown to us by Mumford - that such a situation cannot occur in zero characteristic; i.e. $H^1(X, O_X(-C)) \simeq H^1(\bar{X}, O_{\bar{X}}(-\bar{C}))$ over characteristic zero fields.

Chapter III contains two classical theorems by Castelnuovo. These theorems have been dug out of the litterature by L. Gruson. My only effort was to write them down (with Mohan Kumar). The point, in modern language, is to give bounds for Serre's vanishing theorems in cohomology, in terms of the degree of the given curve in $\mathbb{P}^3$. The two results are the following:

If C is a smooth curve in $\mathbb{P}^3$, J its sheaf of ideals and d its degree, then

a) $H^2(\mathbb{P}^3, J(n)) = 0 \qquad n \geq \frac{d-1}{2}$

b) $H^1(\mathbb{P}^3, J(n)) = 0 \qquad n \geq d-2$

The reader who is interested in equations defining a curve canonically embedded may read the version of Saint-Donat [14] of Petri's theorem, in which coupling the above results with some geometric arguments, he gets the complete list of equations of such a curve. (In general they are of degree 2, but here we only get that the degree is less than or equal to three.)

In Chapter IV we give an answer to an old question of Kronecker (and Severi): a local complete intersection curve in affine three space is set theoretically the intersection of two (algebraic) surfaces. We also give the projective version of D. Ferrand: a local complete intersection curve in $\mathbb{P}^3$ is set-theoretically the set of zeroes of a section of a rank two vector bundle. Unfortunately such vector bundles may not be decomposable. The main idea - which is already in [11], example 2.2 - is that if a curve C is "liee" to itself by a complete intersection, then the ideal sheaf of the curve C in O_X is - upto a twist - the dualising sheaf ω_C of C ([11], Remarque 1.5). Starting from that, we construct an extension of O_C by ω_C, with square of ω_C zero, and then a globalisation of a theorem of R. Fossum [3] finishes the proof. The globalisation is harder in the case of D. Ferrand. It must be said that the final conclusion in $\mathbb{A}^3$ has been made possible by Murthy-Towker [9] (and now Quillen-Suslin [12]) theorem on triviality of vector bundles on $\mathbb{A}^3$. Going back to rank two-vector bundles on $\mathbb{P}^3$ we have now three ways of constructing them:

- Horrock's
- Ferrand's
- and by projection of a canonical curve in $\mathbb{P}^3$

It will be interesting to know the relations between these families. We take this opportunity to ask the following question: Can one generalise Gaeta's theorem (for e.g. [11] Theorem 3.2) in the following way:

Is every smooth curve in $\mathbb{P}^3$ liee by a finite number of "liaisons" to a scheme of zeroes of a section of a rank two-vector bundle? *

Or as R. Hartshorne has suggested: "What are the equivalence classes of curves in $\mathbb{P}^3$, modulo the equivalence relation given by "liaison". A start in this direction has been taken by his student, A. Prabhakar Rao (Liaison among curves in Projective 3-space, Ph.D. Thesis [13]). *

I have news from Oslo, saying that L. Gruson and C. Peskine are starting to understand the mysterious chapter III of Halphen's paper [5]. I hope they will publish their results soon. These works and the yet unpublished notes of D. Ferrand on self-liaison would be a good piece of knowledge on curves in 3-space.

N. Mohan Kumar has written these notes and it is a pleasure for me to thank him for his efficiency, his remarks and his talent to convert the "franglais" I used during the course to "good English". The reader should consider all the "gallissisms" as mine and the "indianisms" him. It has been a real pleasure for me to work with him and to drink beer with him in Bombay - a city which

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* these questions have now been answered by A.P. Rao (the first negatively) in his paper : "Liaisons among curves in $\mathbb{P}^3$," Inventiones Math. 1978.

goes far beyond all that I had expected, in good and in bad. I thank the many people there who gave me the opportunity of living in India and also made my stay enjoyable - R. Sridharan, M.S. Narasimhan, R.C. Cowsik, S. Ramanan and surprisingly Okamoto from Hiroshima University. The typists of the School of Mathematics have typed these manuscripts with care and I thank them very much. I also thank Mathieu for correcting the orthographic mistakes and Rosalie - Lecan for the documentation she helped me with.

CONTENTS

CHAPTER I

PRELIMINARIES

IN THIS CHAPTER, which we offer as an introduction, one will not find many proofs. The aim is to state clearly some concepts so that we can speak rigorously of the different ideals defining a projective embedded variety. We give also the duality theorems and some of their consequences (finiteness, vanishing and Riemann-Roch theorem for curves), notions which play the role of 'Completeness of a linear system' or 'Specialness of a divisor'. The reader will find complete proofs of two intersecting facts:

(i) There exists a curve in $\mathbb{P}^3$ with no imbedded smooth deformation.

(ii) Every curve in $\mathbb{P}^3$ which is locally a complete intersection can be defined by four equations.

For simplicity we throughout assume that the base field is algebraically closed.

A <u>graded ring</u> A is a ring of the form:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots\ldots ,$$

such that A_0 is a ring, A_i's are all A_0-modules and $A_i . A_j \subset A_{i+j}$. Any $f \in A_i$ for some i is said to be a <u>homogeneous element</u> of A. An ideal I of A is said to be a graded ideal, if $\sum f_i \in I$, with $f_i \in A_i$ (i.e. f_i homogeneous) then $f_i \in I$.

Assume A_0 is a field and also A is generated by A_1 over A_0. A_1 a finite dimensional vector space over A_0. We define X = Proj A as follows: Set theoretically $X = \{$ All graded prime ideals of $A \neq A_1 \oplus A_2 \oplus \ldots \}$.

We will give X a scheme structure, by covering X by affine open sets:

Let $f \in A_1$. Then,

$$A_f = (A_f)_0[T, T^{-1}], \qquad (*)$$

where $(A_f)_0 = \{ \frac{g}{f^n} \, g \in A_n \}$. (degree 0 elts. in A_f.)

$(A_f)_0$ is clearly a ring with identity.

(*) is got by mapping T to f and T^{-1} to f^{-1} in A_f. Denote by X_f the set $\{ \mathfrak{p} \in X / f \notin \mathfrak{p} \}$, clearly there is a canonical bijection

$$X_f \longleftrightarrow \text{Spec } (A_f)_0.$$

Transferring the scheme structure to X_f and verifying that this structure is compatible as we vary $f \in A_1$, we get a scheme structure on X.

Example: 1. Let $A = K[X_0, X_1, \ldots, X_n]$ be polynomial ring in $n+1$ variables graded in the natural way: $A_0 = k$. A_1 = vector space of dimension $n+1$ with $X_0, \ldots, X_n$ as generators i.e. A_1 = set of all homogeneous linear polynomials in X_i's. A_n = set of all homogeneous polynomials in

X_i's of deg n.

Then Proj $A = \mathbb{P}^n$, the projective space of dimension n.

2. Let I be any ideal of A generated by homogeneous polynomials $\{f_1, \ldots, f_n\}$. Then $A' = A/I$ is a graded ring $X = \text{Proj } A'$ and Proj $A = \mathbb{P}^n$; X is the closed subvariety of $\mathbb{P}^n$ defined by equations $(f_1, \ldots, f_n)$.

$M = \bigoplus_{n \in \mathbb{Z}} M_n$ is said to be a graded A-module over the graded ring $A = \bigoplus_{i \geq 0} A_i$ if M is an A-module and $A_i . M_n \subset M_{n+i}$.

If M is a graded A-module we can associate a sheaf $\widetilde{M}$ to M over X = Proj A as follows: Over X_f we define the sheaf to be $(M_f)_0$ where $(M_f)_0$

is the set of degree zero elements of M_f. It is a module over $(A_f)_0$. [Recall that X_f = Spec $(A_f)_0$]. One can check that this defines a sheaf over X.

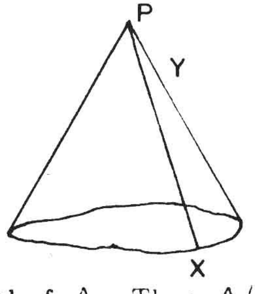
REMARK: $\tilde{A} = O_X$.

If M is a graded A-module, we define M(n) to be the graded A-module given by, $M(n)_k = M_{n+k}$. We denote $\widetilde{M(n)}$ by $\tilde{M}(n)$. In particular,

$$\widetilde{A(n)} = \tilde{A}(n) = O_X(n).$$

If F is any sheaf on X, we denote by F(n) the sheaf $F \otimes_{O_X} O_X(n)$. If X = Proj A, then Y = Spec A is defined to be a cone over X.

Let P denote the point in Y corresponding to the special maximal ideal $A_1 \oplus A_2 \oplus \ldots$ P is defined as the vertex of the cone Y over X.



Let I be any graded ideal of A. Then A/I is a graded ring. Denote by Z, the scheme Proj A/I. It can be easily checked that the canonical map A $\longrightarrow$ A/I induces a closed immersion Z $\longrightarrow$ X = Proj A. Conversely given a closed subscheme Z of X, we can find a graded ideal $I \subset A$, such that the canonical map Proj A/I $\longrightarrow$ X is an isomorphism of Proj A/I with Z. Then we say that I ideally defines Z in X. But this I is not completely determined by Z. One can check that if I is any ideal defining Z, then so does Im^n where m corresponds to the special maximal ideal of A. (i.e. it corresponds to the vertex of the given cone)

Since we are assuming that $A_0 = k$ is a field and A is generated by A_1 over k, where A_1 is finite dimensional over k, we have a graded ring homomorphism,

$$R = k[X_0, X_1, \ldots X_n] \longrightarrow A,$$

which is surjective. [Polynomial rings have the canonical grading]. The kernel is a graded ideal J in R. So we have a closed immersion
$X = \text{Proj } A \hookrightarrow \text{Proj } R = \mathbb{P}^n_k$.

Thus all the schemes we have considered are closed subschemes in some $\mathbb{P}^n_k$ (In particular they are all projective).

<u>REMARK.</u> We have already seen that J need not be unique. But if X is reduced and if we insist that R/J is also reduced then J is unique. [Take J = root ideal of any ideal defining X].

If $X = \text{Proj } R/J$, i.e. J is some ideal of R defining X then using (*) one can verify that $\text{Spec } R/J - [P]$ is uniquely determined. In other words any ideal J which defines X ideally determines the corresponding cone everywhere except the vertex.

<u>Examples:</u>

1. Let $R = k[X_0, X_1]$. So $\text{Proj } R = \mathbb{P}^1_k$.

Let $J_1 = (X_0)$ and $J_2 = (X_0^2, X_0X_1)$. Then $\text{Proj } R/J_1 \cong \text{Proj } R/J_2$. $J_1 \supsetneq J_2$. Note that $(R/J_1)_P$ is Cohen-Macaulay ($\because$ depth $(R/J_1)_P = 1$) and depth $(R/J_2)_P = 0$, where P is the vertex.

2. Take the imbedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ given by :
$(x_0, x_1) \longrightarrow (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$. Then an ideal defining the image in $\mathbb{P}^3$ is $J = (X_0X_3 - X_1X_2, X_0X_2 - X_1^2, X_1X_3 - X_2^2.)$ We see that the variety is not a complete intersection and the vertex of the cone is also not a complete intersection.

We will show now how properties of the vertex affect the variety itself.

PROPOSITION 1.1. Let $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ be a graded ring where A_1 is a finitely generated vector space over k generating A as a graded k-algebra. Let P be the vertex. Then

i) A_P is $R_i \Longrightarrow$ Proj A is R_i

ii) A_P is $S_i \Longrightarrow$ Proj A is S_i

iii) A_P is a complete intersection $\Longrightarrow$ Proj A is locally completely intersection

iv) A_P is a U.F.D $\Longrightarrow A$ is factorial

Proof. Assume that A_P has #. Let # denote any of the properties (i), (ii), (iii). Since Proj A is covered by open sets of the type Spec $A_{(f)_o}$, $f \in A_1$, it suffices to prove that # holds for each one of them.

So let $p \in \operatorname{Spec} A_{(f)_o}$. We want to show that $(A_{(f)_o})p$ has # But $(A_{(f)_o})_p$ has # $\Longleftrightarrow A_{(f)_o}[T, T^{-1}]_{p[T,T^{-1}]}$ has #.

As we have already seen $A_{(f)_o}[T, T^{-1}] \simeq A_f$ and then $p[T, T^{-1}]$ will correspond to a prime ideal qA_f, ($q \hookrightarrow A$ a homogeneous prime ideal.) So it suffices to show that # holds for $A_{f(qA_f)}$. Now q is contained in P, since q is homogeneous and $A_{f(qA_f)} \simeq A_{P(qA_P)}$. The result then follows from the fact that A_P has # and hence any localization of A_P also has #. As iv) is easy we leave it to the reader.

2. COHOMOLOGY OF COHERENT SHEAVES: Let $R = k[X_0, \ldots, X_n]$. Then to any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, one can associate a graded R-module F of finite type. This correspondence is not unique. But given a graded R-module F, we can associate to it a unique sheaf on $\mathbb{P}^n$. For the definition of cohomolo and results on cohomology we refer the reader to FAC by J. P. Serre and loca cohomology by A. Grothendieck. We denote by $O_{\mathbb{P}^n}(1)$ the line bundle got by hyperplane in $\mathbb{P}^n$.

REMARK. $O_{\mathbb{P}^n}(-1) = \mathrm{Hom}\,(O_{\mathbb{P}^n}(1), O_{\mathbb{P}^n}) = (O_{\mathbb{P}^n}(1))^{v}$.

PROPOSITION 2.1. There is an exact sequence for any graded R-module M

$$0 \longrightarrow H_P^0(M) \longrightarrow M \longrightarrow \bigoplus_{m \in \mathbb{Z}} H^0(\mathbb{P}^n, \tilde{M}(m)) \longrightarrow H_P^1(M) \longrightarrow 0,$$

and

$$H_P^{i+1}(M) = \bigoplus_{m \in \mathbb{Z}} H^i(\mathbb{P}^n, \tilde{M}(m)) \quad i \geq 1.$$

Proof. This statement is almost the same as Prop. 2.2 in LC. Putting X = Spec R and P = Y in that result we get

$$0 \longrightarrow H_P^0(M) \longrightarrow M \longrightarrow H^0(\mathrm{Spec}\ R - \{P\}, \tilde{M}) \longrightarrow H_P^1(M) \longrightarrow 0,$$

where $\tilde{M}$ is the sheaf defined by M on Spec R-P and

$$H_P^{i+1}(M) \cong H^i(\mathrm{Spec}\ R - P, \tilde{M}), \quad i > 0.$$

So we only have to check that $H^i(\mathrm{Spec}\ R - P, \tilde{M}) \cong \bigoplus_m H^i(\mathbb{P}^n, M(m))$, for every i, canonically. We have a map $\mathrm{Spec}\ R - P \xrightarrow{p} \mathbb{P}^n$ which is a surjection and an affine map. So

$$H^i(\mathbb{P}^n, p_* \tilde{M}) \longrightarrow H^i(\mathrm{Spec}\ R - P, \tilde{M}).$$

So we want to show that,

$$H^i(\mathbb{P}^n, p_* \tilde{M}) = \bigoplus_m H^i(\mathbb{P}^n, M(m)), \quad \forall i .$$

But one checks that

$$p_* M = \bigoplus_m \tilde{M}(m)$$

canonically and the result follows.

Example: $\tilde{M} = \mathcal{O}_C$, where C is a reduced curve in $\mathbb{P}^n$. $M = R/J$. J is an ideal defining C.

$$R/J = \bigoplus H^0(\mathbb{P}^n, R/J(m)) = \bigoplus H^0(\mathbb{P}^n, \mathcal{O}_C(m))$$

where $\mathcal{O}_C(m) = \mathcal{O}_C(1)^{\otimes m}$ and $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^n}(1)/C$

$$\bigoplus H^0(\mathbb{P}^n, \mathcal{O}_C(m)) = \bigoplus H^0(C, \mathcal{O}_C(m)).$$

Claim: The map $R/J \longrightarrow \bigoplus H^0(C, \mathcal{O}_C(m))$ is injective if J is the biggest ideal defining C.

From the above exact sequence, we get

if $R/J \longrightarrow \bigoplus H^0(C, \mathcal{O}_C(m))$ is injective

then $H^0_P(R/J) = 0$ i.e. $\text{depth}_P R/J \geq 1$. (By Theorem 3.8 of LC.)

then P is not an imbedded component

hence J is the biggest ideal defining C.

Claim: If C is a smooth curve,

$$R/J \longrightarrow \bigoplus_m H^0(C, \mathcal{O}_C(m))$$

is surjective if and only if C is arithmetically normal.

By the above exact sequence

$$R/J \longrightarrow \oplus H^0(C, O_C(m)) \text{ is injective and surjective}$$

$$H^0_P(R/J) = H^1_P(R/J) = 0 \quad \text{depth}_P R/J \geq 2 \text{ by Th. 3.8 of LC}$$

since Spec $R/J - [P]$ is normal we have to check normality only at P. Since P is of codim 2 in Spec R/J, $(R/J)_P$ is normal by Serre's criterion.

i.e. C is arithmetically normal

3. VANISHING THEOREM AND DUALITY

VANISHING THEOREM (SERRE). Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ Then for all $i > 0$, $H^i(\mathbb{P}^n, \mathcal{F}(m)) = 0$ and $H^0(\mathbb{P}^n, \mathcal{F}(m))$ generate $\mathcal{F}(m)$ as $O_{\mathbb{P}^n}$ module for $m \gg 0$.

DUALITY THEOREM. Let $\mathcal{F}$ be a locally free sheaf on $\mathbb{P}^n$, of finite type. $\omega_{\mathbb{P}^n} = \overset{n}{\Lambda}\, \Omega_{\mathbb{P}^n/k}$, where $\Omega^1_{\mathbb{P}^n/k}$ is the sheaf of differentials. So $\omega_{\mathbb{P}^n} = O_{\mathbb{P}^n}(-n-1)$. Then $H^i(\mathbb{P}^n, \mathcal{F}) \times H^{n-i}(\mathbb{P}^n, \mathcal{F} \otimes \omega) \longrightarrow H^n(\mathbb{P}^n, \omega) \simeq k$ is a perfect pairing.

DUALITY ON A LOCALLY COHEN-MACAULAY CURVE C: Let $\mathcal{F}$ be a locally sheaf of finite rank on $C \hookrightarrow \mathbb{P}^n$. Then,

$$H^i(C, \mathcal{F}) \times H^{1-i}(C, \mathcal{F}^{v} \otimes \omega_C) \xrightarrow{\sim} H^1(C, \omega_C) \text{ is a perfect pairing,}$$

$$\text{with } \omega_C = \underline{\mathrm{Ext}}^{n-1}_{\mathbb{P}^n}(O_C, \omega_{\mathbb{P}^n}).$$

1. If X is smooth, $\omega_X = \overset{\max}{\Lambda}\, \Omega^1_{X/k}$.

2. Let X and Y be equidimensional locally Cohen-Macaulay varieties with $X \hookrightarrow Y$. If c is the codimension of X in Y, then

$$\omega_X = \underline{\mathrm{Ext}}^c(O_X, \omega_Y).$$

COROLLARY: If X and Y are as above with X a divisor on Y, then $(L \otimes \omega_Y)|_X = \omega_X$ where L is the line bundle associated to the divisor X.