

LECTURE NOTES IN LOGIC

STABLE DOMINATION AND  
INDEPENDENCE IN ALGEBRAICALLY  
CLOSED VALUED FIELDS

DEIRDRE HASKELL  
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*Stable Domination and Independence in  
Algebraically Closed Valued Fields*

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## **Stable Domination and Independence in Algebraically Closed Valued Fields**

This book addresses a gap in the model-theoretic understanding of valued fields that has, until now, limited the interactions of model theory with geometry. It contains significant developments in both pure and applied model theory.

Part I of the book is a study of stably dominated types. These form a subset of the type space of a theory that behaves in many ways like the space of types in a stable theory. This part begins with an introduction to the key ideas of stability theory for stably dominated types. Part II continues with an outline of some classical results in the model theory of valued fields and explores the application of stable domination to algebraically closed valued fields. The research presented here is made accessible to the general model theorist by the inclusion of the introductory sections of each part.

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LECTURE NOTES IN LOGIC

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## PREFACE

Valuations are among the fundamental structures of number theory and of algebraic geometry. This was recognized early by model theorists, with gratifying results: Robinson's description [45] of algebraically closed valued fields as the model completion of the theory of valued fields; the Ax-Kochen, Ershov study of Henselian fields of large residue characteristic with the application to Artin's conjecture [1, 2, 3, 10]; work of Denef and others on integration; and work of Macintyre, Delon, Prestel, Roquette, Kuhlmann, and others on  $p$ -adic fields and positive characteristic. The model theory of valued fields is thus one of the most established and deepest areas of the subject.

However, precisely because of the complexity of valued fields, much of the work centers on quantifier elimination and basic properties of formulas. Few tools are available for a more structural model-theoretic analysis. This contrasts with the situation for the classical model complete theories, of algebraically closed and real closed fields, where stability theory and o-minimality make possible a study of the category of definable sets. Consider for instance the statement that fields interpretable over  $\mathbb{C}$  are finite or algebraically closed. Quantifier elimination by itself is of little use in proving this statement. One uses instead the notion of  $\omega$ -stability; it is preserved under interpretation, implies a chain condition on definable subgroups, and, by a theorem of Macintyre,  $\omega$ -stable fields are algebraically closed. With more analysis, using notions such as generic types, one can show that indeed every interpretable field is finite or definably isomorphic to  $\mathbb{C}$  itself. This method can be extended to differential and difference fields. Using a combination of such methods and of ideas of manifolds and Lie groups in a definable setting, Pillay was able to prove similar results for fields definable over  $\mathbb{R}$  or  $\mathbb{Q}_p$ . But just a step beyond, a description of interpretable fields seems out of reach of the classical methods. When  $p$ -adic or valuative geometry enters in an essential way, an intrinsic analog of the notion of generic type becomes necessary.

For another example, take the notion of connectedness. In stability, no topology is given in advance, but one manages to define stationarity of types or connectedness of definable groups, by looking at the type space. In o-minimality, natural topologies on definable sets exist, and connectedness,

defined in terms of *definable paths*, is a central notion. In valued fields, the valuation topology is analogous to the o-minimal order topology, and one has the linearly ordered value group that may serve as the domain of a path; but every continuous definable map from the value group to the field is constant, and a model-theoretic definition of connectedness is missing. The lack of such structural model-theoretic understanding of valued fields is a central obstacle to a wider interaction of model theory with geometry in general.

It is this gap that the present monograph is intended to address. We suggest an approach with two components. We identify a certain subset of the type space, the set of *stably dominated types*, that behaves in many ways like the types in a stable theory. Since these types are not literally stable, it is necessary to first describe abstractly an extension of stability theory that includes them. Secondly, we note the existence of o-minimal families of stably dominated types and show, at least over sufficiently rich bases, that any type can be viewed as a limit of such a family. This requires imaginaries in a concrete form, serving as canonical bases of stably dominated types, and a theory of definable maps from  $\Gamma$  into such sets of imaginaries. Thus, whereas type spaces work best in stable theories, and definable sets and maps in o-minimal theories, we suggest here an approach mixing the two. As both the method and the intended applications depend heavily on imaginary elements, we develop some techniques for dealing with them, including prime models that often allow a canonical passage from imaginary to real bases.

We work throughout with the model completion, algebraically closed valued fields. This is analogous to Weil's program of understanding geometry first at the level of algebraic closure. One can hope that the geometry of other valued fields could also, with additional work, be elucidated by this approach. As an example of this viewpoint, consider the known elimination of quantifiers for Henselian fields of residue characteristic zero, relative to the value group and residue field. This was originally derived as an independent theorem. But it is also an immediate consequence of a fact about algebraically closed valued fields of characteristic zero, namely that over any subfield  $F$ , any  $F$ -definable set is definably isomorphic to pullbacks of definable subsets of the residue field and value group. See [19] for more details on this short argument. It is also noted there that definable sets in  $F^n$  can be fibered over the residue field and value group in fibers that are ACVF-definable, and a similar statement for definable bijections between them can be made. Another example is the elimination of imaginaries for  $\mathbb{Q}_p$ , proved in [20] using the ACVF methods of invariant types. We illustrate the phenomenon in Chapter 16 using the theory of  $\mathbb{C}(t)$ .

We restrict ourselves in this monograph to laying the foundations of this approach. Definable maps from  $\Gamma$  into imaginaries were described in [12]; we concentrate here on the theory of stable domination. Only future work based

on these foundations can show to what extent they are successful. We note here that Pillay has defined an analogue of stable domination, *compact domination*, that appears to be useful for thinking about o-minimal groups (cf. [21]). Some progress has been made with the analysis of definable groups using the present methods; a notion of *metastability* has been abstracted (Definition 4.11), and results obtained for Abelian groups in a general metastable setting, and linear groups interpretable in ACVF; see [15]. And in very recent work of one of the authors with Loeser, connections with the Berkovich theory of rigid analytic spaces are beginning to emerge.

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This work has benefitted from discussions over ten years with many mathematicians, and it is hard to single out individuals. The research of Haskell was supported by NSF and NSERC grants, and that of Hrushovski was partially conducted while he was a Clay Mathematics Institute Prize Fellow and was also partially supported by the Israel Science Foundation. The work of Macpherson was supported by EPSRC and LMS grants. All three authors were supported by NSF funding for parts of a semester at the Mathematical Sciences Research Institute, Berkeley.



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## CHAPTER 1

### INTRODUCTION

As developed in [49], stability theory is based on the notion of an *invariant* type, more specifically a *definable type*, and the closely related theory of *independence of substructures*. We will review the definitions in Chapter 2 below; suffice it to recall here that an (absolutely) invariant type gives a recipe yielding, for any substructure  $A$  of any model of  $T$ , a type  $p|A$ , in a way that respects elementary maps between substructures; in general one relativizes to a set  $C$  of parameters, and considers only  $A$  containing  $C$ . Stability arose in response to questions in pure model theory, but has also provided effective tools for the analysis of algebraic and geometric structures. The theories of algebraically and differentially closed fields are stable, and the stability-theoretic analysis of types in these theories provides considerable information about algebraic and differential-algebraic varieties. The model companion of the theory of fields with an automorphism is not quite stable, but satisfies the related hypothesis of simplicity; in an adapted form, the theory of independence remains valid and has served well in applications to difference fields and definable sets over them. On the other hand, such tools have played a rather limited role, so far, in o-minimality and its applications to real geometry.

Where do valued fields lie? Classically, local fields are viewed as closely analogous to the real numbers. We take a “geometric” point of view however, in the sense of Weil, and adopt the model completion as the setting for our study. This is Robinson’s theory ACVF of algebraically closed valued fields. We will view valued fields as substructures of models of ACVF. Moreover, we admit other substructures involving imaginary elements, notably codes for lattices; these have been classified in [12]. This will be essential not only for increasing the strength of the statements, but even for formulating our basic definitions.

A glance at ACVF reveals immediately a stable part, the residue field  $k$ ; and an o-minimal part, the value group  $\Gamma$ . Both are stably embedded, and have the induced structure of an algebraically closed field, and an ordered divisible abelian group, respectively. But they amount between them to a small part of the theory. For instance, over the uncountable field  $\mathbb{Q}_p$ , the residue field has only finitely many definable points, and both  $k$  and  $\Gamma$  are countable in the

model  $\mathbb{Q}_p^a$ . As observed by Thomas Scanlon [46], ACVF is not stable over  $\Gamma$ , in the sense of [48].

We seek to show nevertheless that stability-theoretic ideas can play a significant role in the description of valued fields. To this end we undertake two logically independent but mutually motivating endeavors. In Part I we introduce an extension of stability theory. We consider theories that have a stable part, define the notion of a *stably dominated* type, and study its properties. The idea is that a type can be controlled by a very small part, lying in the stable part; by analogy, (but it is more than an analogy), a power series is controlled, with respect to the question of invertibility for instance, by its constant coefficient. Given a large model  $\mathcal{U}$  and a set of parameters  $C$  from  $\mathcal{U}$ , we define  $\text{St}_C$  to be a many-sorted structure whose sorts are the  $C$ -definable stably embedded stable subsets of the universe. The basic relations of  $\text{St}_C$  are those given by  $C$ -definable relations of  $\mathcal{U}$ . Then  $\text{St}_C(A)$  (the stable part of  $A$ ) is the definable closure of  $A$  in  $\text{St}_C$ . We write  $A \downarrow_C^d B$  if  $\text{St}_C(A) \downarrow \text{St}_C(B)$  in the stable structure  $\text{St}_C$  and  $\text{tp}(B/C \text{St}_C(A)) \vdash \text{tp}(B/CA)$ , and say that  $\text{tp}(A/C)$  is *stably dominated* if, for all  $B$ , whenever  $\text{St}_C(A) \downarrow \text{St}_C(B)$ , we have  $A \downarrow_C^d B$ . In this case  $\text{tp}(A/\text{acl}(C))$  lifts uniquely to an  $\text{Aut}(\mathcal{U}/\text{acl}(C))$ -invariant type  $p$ . Base-change results (under an extra assumption of existence of invariant extensions of types) show that if  $p$  is also  $\text{Aut}(\mathcal{U}/\text{acl}(C'))$ -invariant then  $p|_{C'}$  is stably dominated; hence, under this assumption, stable domination is in fact a property of this invariant type, and not of the particular base set. We formulate a general notion of domination-equivalence of invariant types (2.2). In these terms, an invariant type is stably dominated iff it is domination-equivalent to a type of elements in a stable part  $\text{St}_C$ .

Essentially the whole forking calculus becomes available for stably dominated types. Properties such as definability, symmetry, transitivity, characterization in terms of dividing, lift easily from  $\text{St}_C$  to  $\downarrow^d$ . Others, notably the descent part of base change, require more work and in fact an additional assumption: that for any algebraically closed substructure  $C \subseteq M \models T$ , any type  $p$  over  $C$  extends to an  $\text{Aut}(M/C)$ -invariant type  $p'$  over  $M$ .

We isolate a further property of definable types in stable theories. Two functions are said to have the same *germ* relative to an invariant type  $p$  if they agree generically on  $p$ . In the o-minimal context, an example of this is the germ at  $\infty$  of a function on  $\mathbb{R}$ . Moving from the function to the germ one is able to abstract away from the artifacts of a particular definition. In stability, this is an essential substitute for a topology. For instance, if  $f$  is a function into a sort  $D$ , one shows that the germ is internal to  $D$ ; this need not be the case for a code for the function itself. In many stable applications, the strength of this procedure depends on the ability to reconstruct a representative of the germ from the germ alone. We say that a germ is *strong* if this is the case.

It is easy to see the importance of strong germs for the coding of imaginaries. One wants to code a function; as a first approximation, code the germ of the function; if the code is strong, one has succeeded in coding at least a (generic) piece of the function in question. If it is not, one seems to have nothing at all.

We show that germs of stably dominated invariant types are always strong. The proof depends on a combinatorial lemma saying that finite set functions on pairs, with a certain triviality property on triangles, arise from a function on singletons; in this sense it evokes a kind of primitive 2-cohomology, rather as the fundamental combinatorial lemma behind simplicity has a feel of 2-homology. Curiously, both can be proved using the fundamental lemma of stability.

In [15] it is shown that stable domination works well with definable groups. A group  $G$  is called generically metastable if it has a translation invariant stably dominated definable type. In this case there exists a unique translation invariant definable type; and the stable domination can be witnessed by a definable homomorphism  $h: G \rightarrow H$  onto a connected stable definable group. Conversely, given such a homomorphism  $h$ ,  $G$  is generically metastable iff the fiber of  $h$  above a generic element of  $H$  is a complete type. Equivalently, for any definable subset  $R$  of  $G$ , the set  $Y$  of elements  $y \in H$  such that  $h^{-1}(y)$  is neither contained in, nor disjoint from  $R$  is a small set; no finite union of translates of  $Y$  covers  $H$ . We show this in Theorem 6.13, again using strong germs.

The general theory is at present developed locally, at the level of a single type. It is necessary to say when we expect it to be meaningful globally. The condition cannot be that every type be stably dominated; this would imply stability. Instead we would like to say that uniformly definable families of stably dominated types capture, in some sense, all types. Consider theories with a distinguished predicate  $\Gamma$ , that we assume to be linearly ordered so as to sharply distinguish it from the stable part. We define a theory to be *metastable over*  $\Gamma$  (Definition 4.11) if every type over an algebraically closed set extends to an invariant type, and, over sufficiently rich base sets, every type falls into a  $\Gamma$ -parameterized family of stably dominated types. We show that this notion is preserved under passage to imaginary sorts.

The proviso of “sufficiently rich base set” is familiar from stability, where the primary domination results are valid only over sufficiently saturated models; a great deal of more technical work is then needed to obtain some of them over arbitrary base. The saturation requirement (over “small” base sets) is effective since types over a model are always based on a small set. In the metastable context, more global conditions incompatible with stability are preferred. This will be discussed for ACVF below.

For some purposes, extensions of the base are harmless and the theory can be used directly. This is so for results asserting the existence of a canonical

definable set or relation of some kind, since a posteriori the object in question is defined without extra parameters. This occurred in the classification of maps from  $\Gamma$  in [12]. Another instance is in [15], where under certain finiteness of rank assumptions, it is shown that a metastable Abelian group is an extension of a group interpretable over  $\Gamma$  by a definable direct limit of generically metastable groups.

In Part II we study ACVF. This is a  $C$ -minimal theory, in the sense of [35], [13]: there exists a uniformly definable family of equivalence relations, linearly ordered by refinement; their classes are referred to as (ultrametric) *balls*; and any definable set (in 1-space) is a Boolean combination of balls. In strongly minimal and  $o$ -minimal contexts, one often argues by induction on dimension, fibering an  $n$ -dimensional set over an  $n - 1$ -dimensional set with 1-dimensional fibers, thus reducing many questions to the one-dimensional case over parameters. This can also be done in the  $C$ -minimal context. Let us call this procedure “*dévisage*”.

A difficulty arises: many such arguments require canonical parameters, not available in the field sort alone. And certainly all our notions, from algebraic closure to stable embeddedness, must be understood with imaginaries. The imaginary sorts of ACVF were given concrete form in [12]: the spaces  $S_n$  of  $n$ -dimensional lattices, and certain spaces  $T_n$ , fibered over  $S_n$  with fibers isomorphic to finite dimensional vector spaces over the residue field. But though concrete, these are not in any sense one-dimensional: attempting to reduce complexity by induction on the number of coordinates only leads to subsets of  $S_n$ , which is hardly simpler than  $(S_n)^m$ .

Luckily,  $S_n$  itself admits a sequence of fibrations  $S_n = X_N \rightarrow X_{N-1} \rightarrow \dots \rightarrow X_0$ , with  $X_0$  a point and such that the fibers of  $X_{i+1} \rightarrow X_i$  are  $o$ - or  $C$ -minimal. This uses the transitive action of the solvable group of upper triangular matrices on  $S_n$ ; see the paragraph following Proposition 7.14. There is a similar statement for  $T_n$  (where strongly minimal fibers also occur.) It follows that any definable set of imaginaries admits a sequence of fibrations with successive fibers that are strongly,  $o$ - or  $C$ -minimal (“unary sets”), or finite. *Dévisage* arguments are thus possible.

One result obtained this way is the existence of invariant extensions. A type over a base set  $C$  can only have an invariant extension if it is *stationary*, i.e., implies a complete type over  $\text{acl}(C)$ . We show that in ACVF, every stationary type over  $C$  has an  $\text{Aut}(\mathcal{U}/C)$ -invariant extension. For  $C$ -minimal sets (including strongly minimal and  $o$ -minimal ones), there is a standard choice of invariant extension: the extension avoiding balls of radius smaller than necessary.

But this does not suffice to set up an induction, since for finite sets there is no invariant extension at all. Thus a minimal step of induction consists of *finite covers of  $C$ -minimal sets*, i.e., with sets  $Y$  admitting a finite-to-one map

$\pi: Y \rightarrow X$ , with  $X$  unary. This is quite typical of ACVF, and resembles algebraic geometry, where dévissage can reduce as far as *curves* but not to a single variable. In the o-minimal case, by contrast, one can do induction on ambient dimension, or the number of coordinates of a tuple; this explains much of the more “elementary” feel of basic o-minimality vs. strong minimality.

The additional ingredient needed to obtain invariant extensions of types is the *stationarity lemma* from [12], implying that if  $\pi$  admits a section over a larger base, then it admits a section over  $\text{acl}(C)$ . See Lemma 8.10. For the theory ACF over a perfect field, stationarity corresponds to the notion of a regular extension, and the stationarity lemma to the existence of a geometric notion of irreducibility of varieties. It is instructive to recall the proof for ACVF. Given a finite cover  $\pi: Y \rightarrow X$  as above, a section  $s$  of  $Y$  will have a strong germ with respect to the canonical invariant extension of any type of  $X$ . Generic types of closed balls are stably dominated; for these, by the results of Part I, all functions have strong germs. Other types are viewed as limits of definable maps from  $\Gamma$  into the space of generics of closed balls. For instance if  $\tilde{b}$  is an open ball, consider the family of closed sub-balls  $b$  of  $\tilde{b}$ ; these can be indexed by their radius  $\gamma \in \Gamma$  the moment one fixes a point in  $\tilde{b}$ ; by the stably dominated case, one has a section of  $\pi$  over each  $b$ . The classification of definable maps from  $\Gamma$  (actually from finite covers of  $\Gamma$ ) is then used to glue them into a single section, over the original base. This could be done abstractly for  $C$ -minimal theories whose associated (local) linear orderings satisfy  $\text{dcl}(\Gamma) = \text{acl}(\Gamma)$ . The proof of elimination of imaginaries itself has a similar structure; see a sketch at the end of Chapter 15.

Another application of the unary decomposition is the existence of canonical resolutions, or prime models. In the field sorts, ACVF has prime models trivially; the prime model over a nontrivially valued field  $F$  is just the algebraic closure  $F^{\text{alg}}$ . In the geometric sorts the situation becomes more interesting. The algebraic closure does not suffice, but we show that finitely generated structures (or structures finitely generated over models) do admit canonical prime models. A key point is that the prime model over a finitely generated structure  $A$  add to  $A$  no elements of the residue field or value group. This is important in the theory of motivic integration; see the discussion of resolution in [19]. A further application of canonical resolution is a quantifier-elimination for  $\mathbb{C}((t))$  in the  $\mathcal{G}$ -sorts, relative to the value group  $\Gamma$ . In essence resolution is used to produce functions on imaginary sorts; in fact for any  $\mathcal{G}$ -sort represented as  $X/E$  and any function  $h$  on  $X$  into the value group or residue field, there exists a function  $H$  on  $X/E$  such that  $H(u) = h(x)$  for some  $x \in X/E$ .

The construction of prime models combines the decomposition into unary sets with the idea of opacity. An equivalence relation  $E$  on  $X$  is called *opaque* if any definable subset of  $X$  is a union of classes of  $E$ , up to a set contained in finitely many classes. This is another manifestation of a recurring theme.

Given  $f: X \rightarrow Y$  and an ideal  $I$  on  $Y$ , we say  $X$  is dominated by  $Y$  via  $(f, I)$  if for any subset  $R$  of  $X$ , for  $I$ -almost every  $y \in Y$ , the fiber  $f^{-1}(y)$  is contained in  $R$  or is disjoint from it. For stable domination,  $Y$  is stable and  $I$  is the forking ideal; for stationarity,  $f$  has finite fibers, and  $I$  is the dual ideal to an invariant type; for opacity,  $I$  is the ideal of finite sets. The equivalence relations associated with the analyses of  $S_n$  and  $T_n$  above are opaque. For an opaque equivalence relation, all elements in a non-algebraic class have the same type (depending only on the class); this gives a way to choose elements in such a non-algebraic class canonically up to isomorphism. Algebraic classes are dealt with in another way.

We now discuss the appropriate notion of a “sufficiently rich” structure. In the stable part, saturation is the right requirement; this will not actually be felt in the present work, since the stable part is  $\aleph_1$ -categorical and does not really need saturation. For the o-minimal part, a certain completeness condition turns out to be useful; see Chapter 13.2. It allows the description of the semi-group of invariant types up to domination-equivalence, and a characterization of forking in ACVF over very rich bases. For the most part however neither of these play any role; the significant condition is richness over the stable and the o-minimal parts. Here we adopt Kaplansky’s maximally complete fields. An algebraically closed valued is maximally complete if it has no proper immediate extensions. It follows from ([26], [27]) that any model of ACVF embeds in a maximally complete field, uniquely up to isomorphism. Since we use all the geometric sorts, a ‘rich base’ for us is a model of ACVF whose field part is maximally complete.

Over such a base  $C$ , we prove first, using standard results on finite dimensional vector spaces over maximally complete fields, that any field extension  $F$  is dominated by its parts in the residue field  $k(F)$  and the value group  $\Gamma_F$ . This kind of domination does not admit descent. A stronger statement is that  $F$  is dominated by the stable part over  $C$  together with  $\Gamma_F$ , so that the type of any element of  $F^n$  over  $C \cup \Gamma_F$  is stably dominated. After an algebraic interpretation of this statement, it is deduced from the previous one by a perturbation argument. Both these results are then extended to imaginary elements.

We interpret the last result as follows: an arbitrary type lies in a family of stably dominated types, definably indexed by  $\Gamma$ . Note that  $k$  and  $\Gamma$  play asymmetric roles here. Indeed, at first approximation, we develop what can be thought of as the model theory of  $k^\Gamma$ , rather than  $k \times \Gamma$ . However  $k^\Gamma$  is presented by a  $\Gamma$ -indexed system of *opaque* equivalence relations, each hiding the structure on the finer ones until a specific class is chosen. This kind of phenomenon, with hidden forms of  $k^n$  given by finitely many nested equivalence relations, is familiar from stability theory; the presence of a definable directed system of levels is new here.



Even for fields, the stable domination in the stronger statement cannot be understood without imaginaries. Consider a field extension  $F$  of  $C$ ; for simplicity suppose the value group  $\Gamma_F$  of  $F$  is generated over  $\Gamma_C$  by one element  $\gamma$ . There is then a canonical vector space  $V_\gamma$  over the residue field. If  $\gamma$  is viewed as a code for a closed ball  $E_\gamma = \{x : v(x) \geq \gamma\}$ , the elements of  $V_\gamma$  can be taken to be codes for the maximal open sub-balls of  $E_\gamma$ . The vector space  $V_\gamma$  lies in the stable part of the theory, over  $C(\gamma)$ . We show that  $F$  is dominated over  $C(\gamma)$  by elements of  $k(F) \cup V_\gamma(F)$ . Note that  $k(F)$  may well be empty.

Over arbitrary bases, invariant types orthogonal to the value group are shown to be dominated by their stable part; this follows from existence of invariant extensions, and descent.

At this point, we have the metastability of ACVF. We now seek to relate this still somewhat abstract picture more directly with the geometry of valued fields. We characterize the stably dominated types as those invariant types that are orthogonal to the value group (Chapter 8.) In Chapter 14, we describe geometrically the connection between a stably dominated type  $P$  and the associated invariant type  $p$ , when  $P$  is contained in an algebraic variety  $V$ . In the case of ACF, the invariant extension is obtained by avoiding all proper subvarieties. In ACVF, the demand is not only to avoid but to stay as far away as possible from any given subvariety. See Theorem 14.12. In ACF the same prescription yields the unique invariant type of any definable set; it is not necessary to pass through types. In ACVF the picture for general definable sets is more complicated. But for a definable subgroup  $G$  of  $\mathrm{GL}_n(K)$ , or for a definable affine homogeneous space, we show that a translation invariant stably dominated type is unique if it exists, and that in this case it is again the type of elements of maximal distance from any proper subvariety of the Zariski closure of  $G$ .

In chapter 15 the ideas are similar, but the focus is on canonical bases. Any definable type, in general, has a smallest substructure over which it is defined. In ACF, this is essentially the field of definition of the associated prime ideal. We obtain a similar geometric description for stably dominated types; the ideal of regular functions vanishing on the type is replaced by the  $R$ -module of functions taking small values on it.

While presented here for *stably dominated types*, where the theory flows smoothly from the main ideas, within the text we try to work with weaker hypotheses on the types when possible. Over sufficiently rich base structures, all our results can be read off from the main domination results discussed above. But over smaller bases this is not always the case, leading us to think that perhaps a general principle remains to be discovered. An example is the theorem of Chapter 10, that an indiscernible sequence whose canonical base (in an appropriate sense) is orthogonal to  $\Gamma$ , is in fact an indiscernible set, and