lecture notes in pure and applied mathematics

classification theory of semi-simple algebraic groups

I. Satake

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Classification Theory of Semi-Simple Algebraic Groups

Lecture Notes in Pure and Applied Mathematics

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Foreword (added on April, 1971)

This is essentially a photographical reproduction of my Lecture-Notes issued at the University of Chicago in 1967. Taking this opportunity of revision, I tried to make it more readable, eliminating misprints and adding a few foot-notes, a new bibliography, an index of terms, and a list of notations. I also included an Appendix written by M. Sugiura of the University of Tokyo, which gives a very efficient way of classifying real simple algebraic groups in simplification of Araki's method. I should like to express here my gratitude to Sugiura for this invaluable addition to my Notes. My thanks are also due to S. Kobayashi, who invited me to join to the new program of Marcel Dekker mathematics series, to a number of my friends for their kind suggestions for improvements of the Notes, especially to Mrs. Doris Schattschneider for her constant assistance, and finally to Mrs. Laura Hurbace for her fine job in typing these intricate materials.

I. S.

Preface

These notes are based on my course on "Classification-theory of semi-simple algebraic groups" given at the University of Chicago in the winter quarter of 1967. Though its primary aim was to give a general idea of the classification-theory, I thought it convenient to include an outline of the basic theory of algebraic groups, in view of the fact that no standard textbook is as yet available. In this part, proofs are often very sketchy, or completely omitted, but references are given to indicate where a more complete proof is to be found. Thus it is hoped that the graduate student with a sound background in algebra can easily seize the main idea without going into too much detail.

I gratefully acknowledge my debt to Mrs. Doris Schattschneider who kindly helped me in taking notes, reading proofs, and elaborating them in the form presented here.

I. Satake

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I. PRELIMINARIES ON ALGEBRAIC GROUPS

This chapter is an exposition of definitions and known results. The bibliographical references following theorems, or titles of sections indicate where more details and proofs can be found. In a few instances, a proof has been sketched here.

§1. Affine algebraic sets

1.1 Definitions ([4] Chap. II and III; [1] Chap. I, §1)

Notation: Ω : universal domain (i.e., a sufficiently large algebraically closed field)

Ω^N: N-dimensional affine space over Ω

$x = (x_1,\ldots,x_N) = (x_i)$: a point in Ω^N

$\Omega[X] = \Omega[X_1,\ldots,X_N]$: algebra of polynomials in N variables with coefficients in Ω

$\Omega(X)$: quotient field of $\Omega[X]$.

Definition: A subset $A \subset \Omega^N$ is called an (affine) algebraic set if there exists a subset $\mathfrak{M} \subset \Omega[X]$ such that $A = \{x \in \Omega^N \mid f(x) = 0$ for all $f \in \mathfrak{M}\}$. (A is denoted as $A(\mathfrak{M})$, the algebraic set determined by $\mathfrak{M}$; we also write $A \leftarrow \mathfrak{M}$).

Let $A = A(\mathfrak{M})$ be an algebraic set, and put

$$\mathfrak{a}(A) = \{f \in \Omega[X] \mid f(x) = 0 \text{ for all } x \in A\}.$$

Clearly $\mathfrak{a}(A)$ is an ideal in $\Omega[X]$, containing $\mathfrak{M}$. Since $\Omega[X]$ is Noetherian, there exists a finite set of polynomials $f_1,\ldots,f_r$ in $\Omega[X]$ which generate $\mathfrak{a}(A)$; then $A = A(\mathfrak{a}(A)) = A(f_1,\ldots,f_r)$. Thus the correspondence between an algebraic set and its corresponding ideal $\mathfrak{a}(A)$ is one-to-one; we will use the notation $A \longleftrightarrow \mathfrak{a}$ ($\mathfrak{a} = \mathfrak{a}(A)$).

It is easy to see that if A_1 and A_2 are algebraic sets in Ω^N, with

$A_1 \longleftrightarrow \mathfrak{a}_1$, $A_2 \longleftrightarrow \mathfrak{a}_2$, then $A_1 \cup A_2 \longleftrightarrow \mathfrak{a}_1 \cap \mathfrak{a}_2$, and $A_1 \cap A_2 \longleftarrow \mathfrak{a}_1 + \mathfrak{a}_2$ ($\longleftrightarrow$ does not hold for the latter). In general, if $A_\alpha \longleftrightarrow \mathfrak{a}_\alpha$, α running through any set of indices, then $\cap A_\alpha \longleftrightarrow \Sigma\, \mathfrak{a}_\alpha$. Also, if A and B are algebraic sets in Ω^N and Ω^M respectively, with $A \longleftrightarrow \mathfrak{a} \subset \Omega[X]$, $B \longleftrightarrow \mathfrak{b} \subset \Omega[Y]$, then $A \times B$ is an algebraic set in Ω^{N+M} determined by $\mathfrak{a}\Omega[Y] + \mathfrak{b}\Omega[X]$.

<u>Definition</u>: An algebraic set is called <u>irreducible</u> if $A = A_1 \cup A_2$ (A_1, A_2 non-empty algebraic sets) implies $A = A_1$ or A_2. (An irreducible algebraic set is sometimes called a "variety.")

It follows from the remarks above that an algebraic set A is irreducible if and only if $\mathfrak{a}(A)$ is a prime ideal. Also, every algebraic set can be decomposed uniquely as a finite union of irreducible algebraic sets:

$$A = \bigcup_{i=1}^{n} A_i, \quad A_i \text{ irreducible (all } i\text{), and } A_i \not\subset A_j \text{ if } i \neq j.$$

Now, let k be a subfield of Ω. If $\mathfrak{m}$ is a subset of $\Omega[X]$, denote $\mathfrak{m}_k = \mathfrak{m} \cap k[X]$.

<u>Definition</u>: An algebraic set A is <u>k-closed</u> if and only if there exists a subset $\mathfrak{m} \subset k[X]$ such that $A \longleftarrow \mathfrak{m}$. (Equivalently, $A = A(\mathfrak{a}(A)_k)$.)

An algebraic set A is <u>defined over</u> k (we write A/k) if and only if $\mathfrak{a}(A)$ has a basis in $k[X]$. (Equivalently, $\mathfrak{a}(A) = \mathfrak{a}(A)_k \otimes_k \Omega$.)

If A is defined over k, we say that k is a field of definition for A, and A is sometimes called a "k-rational" algebraic set. It is clear from the definition that if A is defined over k, then A is k-closed.

<u>Definition</u>: Let σ be an automorphism of Ω, and A an algebraic set. The <u>conjugate of A</u> by σ is the set $A^\sigma = \{x^\sigma = (x_i^\sigma) \mid x \in A\}$.

The automorphism σ acts on $\Omega[X]$ (by transforming the coefficients of polynomials), and it is clear that if $A \longleftrightarrow \mathfrak{a}$, then $A^{\sigma} \longleftrightarrow \mathfrak{a}^{\sigma}$. If A is k-closed, then A^{σ} depends only on the restriction of σ to k.

Notation: For $k \subset K$, subfields of Ω, let $\bar{k}$ = algebraic closure of k; k^i = inseparable closure of k; k^s = separable closure of k; Aut(K/k) (or Gal(K/k)) the group of automorphisms of K leaving k pointwise fixed.

Proposition 1.1.1: For an algebraic set A, the following conditions are equivalent:

1) A is defined over k^i.

2) A is k-closed.

3) $A^{\sigma} = A$ for all $\sigma \in \mathrm{Aut}(\Omega/k)$.

3') A is $\bar{k}$-closed, and $A^{\sigma} = A$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$.

(1) => 2) => 3) is almost trivial; 3) => 1) follows from the Lemma of Weil on field of definition.)

Corollary: If A is defined over k^s and A is k-closed, then A is defined over k.

From this proposition we see immediately that if k is a perfect field (i.e., $k = k^i$), then the terms "k-closed" and "defined over k" for an algebraic set are synonymous. Later we will only be concerned with the case of k a perfect field.

From the proposition (or the definition), it is easy to check that if A_1, A_2 are k-closed algebraic sets in Ω^N, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are also; in general, if $\{A_\alpha\}$ is any collection of k-closed sets, then $\bigcap A_\alpha$ is k-closed. Thus k-closed algebraic sets satisfy the usual topological conditions of closed sets. The topology on Ω^N having as its closed sets the k-closed algebraic sets is called the "Zariski-k-topology" (or "Zariski topology" when $k = \Omega$).

Unless otherwise specified, in all that follows, by "k-open" (resp., "open") and "k-closed" (resp., "closed") sets in Ω^N, we will always mean with respect to the Zariski-k (resp., Zariski) topology.

It should be noted that the Zariski topology on Ω^N does not satisfy the Hausdorff separation axiom; in fact, if O_1 and O_2 are any non-empty k-open subsets of Ω^N, then $O_1 \cap O_2$ is also a non-empty k-open subset. In addition, any (relatively) open subset of an irreducible set A is necessarily dense in A.

1.2 Rational mappings ([4] Chap. IV; [1] Chap. I, §1)

Definition: Let A be an algebraic set in Ω^N.

A polynomial function (defined over k) on A is the restriction to A of a function defined by a polynomial in $\Omega[X]$ (resp., $k[X]$).

A rational function (defined over k) on A is the restriction to A of a function defined by a rational quotient f/g in $\Omega(X)$ (resp., $k(X)$), with g not vanishing identically on each irreducible component of A. (This last condition is equivalent to: if $A = \cup A_i$ is the decomposition of A into irreducible components, and $A_i \longleftrightarrow \mathfrak{P}_i$, then $g \notin \mathfrak{P}_i$, all i).

Notation: We denote by $\Omega[A]$ (resp., $k[A]$) the ring of polynomial functions on A (defined over k), and by $\Omega(A)$ (resp., $k(A)$) the ring of rational functions on A (defined over k).

The ring $\Omega[A]$ can be canonically identified with $\Omega[X]/\mathfrak{a}(A)$, so that it is an integral domain when A is irreducible, and in that case, $\Omega(A)$ is just the quotient field of $\Omega[A]$.

Definition: Let A be an irreducible algebraic set. The dimension of A is the transcendence degree of the field extension $\Omega(A)/\Omega$. (We write

$\dim A = \dim(\Omega(A)/\Omega)$.)

When A is irreducible, and A/k, one has $\Omega[A] = k[A] \otimes_k \Omega$, so that $\dim A = \dim(k(A)/k)$.

Definition: Let A and B be algebraic sets in Ω^N and Ω^M respectively. A polynomial (resp., rational) map φ from A to B is a mapping given by $\varphi = (\varphi_1, \ldots, \varphi_M)$, $\varphi_i \in \Omega[A]$ (resp., $\varphi_i \in \Omega(A)$), $1 \leq i \leq M$. If φ is a rational map from A to B and each φ_i is represented by $f_i/g_i \in \Omega(A)$, and $x \in A$ satisfies $g_i(x) \neq 0$, $1 \leq i \leq M$, then we say that φ is defined at x, and the value of φ at x is $\varphi(x) = (\varphi_1(x), \ldots, \varphi_M(x)) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_M(x)}{g_M(x)}\right) \in B$. We say that φ is defined over k (we write φ/k) if $\varphi_i \in k[A]$ (resp., $k(A)$), $1 \leq i \leq M$.

From the definitions, we see that a rational function on an algebraic set A is a rational map from A to $\Omega^1 = \Omega$. It also follows that any rational map φ from A to B is defined on a non-empty open set in each irreducible component of A. In fact, if we denote by A_φ the subset of points of A at which φ is defined, then $A_\varphi = \cup A_{i\varphi}$ (A_i the irreducible components of A), and if φ/k, then $A_{i\varphi}$ is a k-open set in A_i (for all i).

Proposition 1.2.1: A rational map φ from A to B is a polynomial map if and only if $A_\varphi = A$.

Definition: We say a polynomial map φ is a birational isomorphism if φ is bijective and φ^{-1} is also a polynomial map.

Notation: If φ is a rational map from A to B, and M is any subset of A, then denote by $\varphi(M)$ the set-theoretic image of $M_\varphi = A_\varphi \cap M$ by φ in B; and denote by $\overline{\varphi(M)}$ the Zariski-closure of $\varphi(M)$ in B. ($\overline{\varphi(M)}$ is called the algebraic image of M by φ.)

If A is a k-closed algebraic set in Ω^N, denote $A_k = A \cap k^N$. (A_k is called the set of k-rational points of A.)

If φ is a rational map from A to B, with φ/k, A and B k-closed, then clearly $\varphi(A_k) \subset B_k$; in particular, if φ is an isomorphism, φ gives a one-to-one correspondence between A_k and B_k. The following proposition sums up some facts relating $\varphi(A)$ and $\overline{\varphi(A)}$.

Proposition 1.2.2: Let A and B be algebraic sets, A irreducible, and φ a rational map from A to B. Then:

1) $\varphi(A)$ contains a set which is relatively open in $\overline{\varphi(A)}$. In fact, if U is any non-empty (relatively) open subset of A, then $\varphi(U)$ contains a subset which is relatively open in $\overline{\varphi(A)}$.
2) $\overline{\varphi(A)}$ is irreducible.
3) If A is defined over k and φ is defined over k, then $\overline{\varphi(A)}$ is defined over k.

If A and B are irreducible algebraic sets, and φ is a surjective rational map from A to B, then there is a natural injection $\Omega(A) \leftarrow \Omega(B)$ given by $\psi\circ\varphi \leftarrow \psi$ ($\psi \in \Omega(B)$). Under this injection, $\Omega(B)$ can be identified with a subfield of $\Omega(A)$, and we make the following definition.

Definition: The degree of φ, denoted deg φ, is the degree $[\Omega(A):\Omega(B)]$, if this is finite; otherwise the degree of φ is zero. We call φ inseparable (resp., separable) if $\Omega(A)/\Omega(B)$ is a purely inseparable (resp., separable) extension.

§2. Affine algebraic groups ([1] Chap. I; [2] exposé 3; [13] Chap. I)

2.1 Definitions

Definition: G is called an (affine) algebraic group if

1) G is an abstract group;

2) G is an algebraic set in Ω^N;

3) The mapping $G \times G \to G$, $(x,y) \to x^{-1}y$ is a polynomial map.

G is <u>defined over</u> k (write G/k) if G as an algebraic set is defined over k, and the mapping in 3) is defined over k.

If G is an algebraic group, then for any fixed a ε G, the left (resp., right) translation

$$L_a: x \to ax, \quad x \in G$$

$$(R_a: x \to xa, \quad x \in G)$$

is an automorphism of G with respect to the structure of an algebraic set. Since left translations are transitive, G is a "homogeneous" algebraic set; in particular, G has no "singular" points. [1] These facts are used in the proofs of some of the properties of algebraic groups.

If G is an algebraic group defined over k, then the identity element of G is k-rational, and it is easily seen that G_k is an abstract group.

If G is an algebraic group and G° is an irreducible component of G containing the identity element, 1, then it can be shown that G° is the <u>only</u> irreducible component of G containing 1. Further, we have:

<u>Proposition 2.1.1</u>: Let G be an algebraic group defined over k, G° the irreducible component of G containing 1. Then G° is a normal, algebraic subgroup of G, defined over k, and $G = \bigcup g_i G^\circ$, the coset decomposition of G with respect to G°, is the decomposition of G into irreducible components.

From this proposition, we see that an algebraic group G is irreducible if and only if it is a connected set in the Zariski topology.

(Note: the words "connected" and "irreducible" are not interchangeable for an arbitrary algebraic set A.) Also from this proposition, we see that the dimension of each of the irreducible components of G is the same as dim G°; thus we have the following

Definition: The dimension of an algebraic group G, denoted dim G, is equal to dim G°.

Examples of algebraic groups

Ex. 1. $G = \mathbb{G}_a = \Omega$, the "additive" group of Ω.

$\mathbb{G}_a$ is defined by the zero polynomial, i.e., $\mathbb{G}_a = A(0)$. dim $\mathbb{G}_a = 1$.

Ex. 2. $G = \mathbb{G}_m \simeq \Omega^*$, the "multiplicative" group of Ω.

$\mathbb{G}_m \subset \Omega^2$, and $\mathbb{G}_m = A(XY - 1)$. dim $\mathbb{G}_m = 1$.

Ex. 3. $G = SL(n)$, the "special linear group."

$SL(n) \subset \Omega^{n^2}$, and $SL(n) = A(\det(X_{ij}) - 1)$.

Ex. 4. $G = GL(n)$, the "general linear group."

$GL(n) \subset \Omega^{n^2+1}$, and $GL(n) = A(\det(X_{ij})Y - 1)$.

All of these groups are connected (since their corresponding ideals, being generated by an irreducible polynomial, are prime), and all are defined over the prime field. Note that when k is a topological field, then the group $GL(n)_k = GL(n,k)$ becomes a topological group with respect to the natural topology on k^{n^2+1}. With respect to this natural topology, it can be shown that $GL(n,\mathbb{C})$ is connected, $GL(n,\mathbb{R})$ has two connected components, and $GL(n,\mathbb{Q}_p)$ is totally disconnected. Thus, the Zariski (k-)topology and the natural topology should be carefully distinguished.

Ex. 5. Let G_1, G_2 be algebraic groups in Ω^N and Ω^M respectively. Then $G_1 \times G_2 \subset \Omega^{N+M}$ is also an algebraic group, and is called the direct