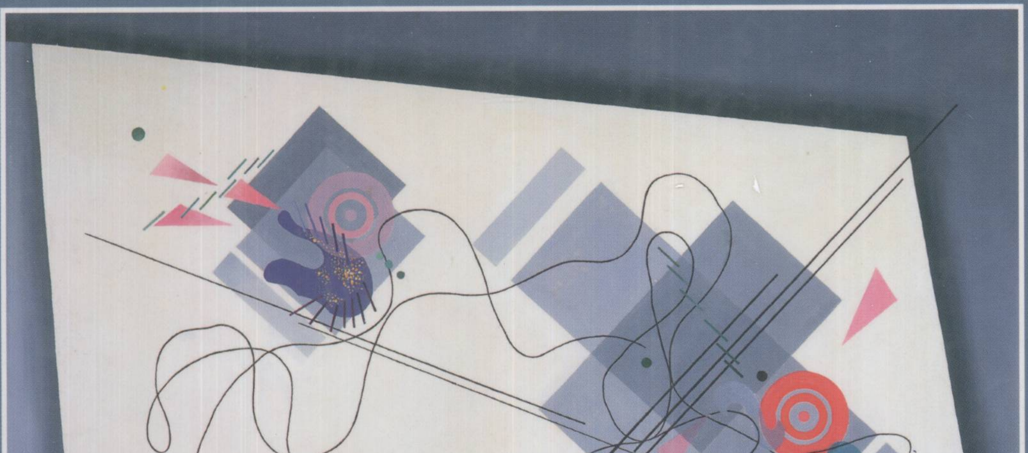
Differential Geometry and Its Applications

Second Edition



John Oprea

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

Second Edition

John Oprea
Cleveland State University

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To my mother and father,
Jeanne and John Oprea.

Preface to First Edition (Revised)

How and what should we teach today's undergraduates to prepare them for careers in mathematically oriented areas? Furthermore, how can we ameliorate the quantum leap from introductory calculus and linear algebra to more abstract methods in both pure and applied mathematics? There *is* a subject which can take students of mathematics to the next level of development and this subject is, at once, intuitive, calculable, useful, interdisciplinary and, *most importantly*, interesting. Of course, I'm talking here about Differential Geometry, a subject with a long, wonderful history and a subject which has found new relevance in areas ranging from machinery design to the classification of four-manifolds to the creation of theories of Nature's fundamental forces to the study of DNA.

Differential Geometry provides the perfect transition course to higher mathematics and its applications. It is a subject which allows students to see mathematics for what it is — not the compartmentalized courses of a standard university curriculum, but a unified whole mixing together geometry, calculus, linear algebra, differential equations, complex variables, the calculus of variations and various notions from the sciences. Moreover, Differential Geometry is not just for mathematics majors, but encompasses techniques and ideas relevant to students in engineering and the sciences. Furthermore, the subject itself is not quantized. By this, I mean that there is a continuous spectrum of results which proceeds from those which depend on calculation alone to those whose proofs are quite abstract. In this way students gradually are transformed from calculators to thinkers.

Into the mix of these ideas now comes the opportunity to visualize concepts and constructions through the use of computer algebra systems such as Maple and Mathematica. Indeed, it is often the case that the consequent visualization goes hand-in-hand with the understanding of the mathematics behind the computer construction. For instance, in Chapter 5, I use Maple to visualize geodesics *on surfaces* and this requires an understanding of the idea of solving a system of differential equations numerically and displaying the solution. Further, in this case, visualization is not an empty exercise in computer technology, but actually clarifies various phenomena such as the bound on geodesics due to the Clairaut relation. There are many other examples of the benefits of computer algebra systems to understanding concepts and solving problems. In particular, the procedure for plotting geodesics can be modified to show equations of motion of particles constrained to surfaces. This is done in Chapter 7 along with describing procedures relevant to the calculus of variations and optimal control. At the end of Chapters 1, 2, 3, 5 and 7 there are sections devoted to explaining how Maple fits into the framework of Differential Geometry. I have tried to make these sections a rather informal tutorial as opposed to just laying out procedures. This is both good and bad for the reader. The good comes from the little tips about pitfalls and ways to avoid them; the bad comes from my personal predelictions and the simple fact that I am not a Maple expert. What you will find in this text is the sort of Maple that anyone can do. Also, I happen to think that Maple is easier for students to learn than Mathematica and so I use it here. If you prefer Mathematica, then you can, without too much trouble I think, translate my procedures from Maple into Mathematica or you can look at [**Gra93**] for a huge number of Mathematica geometry procedures and examples.

In spite of the use of computer algebra systems here, this text is traditional in the sense of approaching the subject from the point of view of the 1800's. What *is* different about this book is that a conscious effort has been made to include material that I feel science and math majors should know. For example, although it is possible to find mechanistic descriptions of phenomena such as Clairaut's relation or Jacobi's theorem and geometric descriptions of mechanistic phenomena such as the precession of Foucault's pendulum in advanced texts (see [**Arn78**] and [**Mar92**]), I believe they appear here for the first time in an undergraduate text. Also, even when dealing with mathematical matters alone, I have always tried to keep some application, whether mathematical or not, in mind. In fact, I think this helps to show the boundaries between physics (e.g. soap films) and mathematics (e.g. minimal surfaces).

This book originally began as an attempt to fashion a one-quarter course in Differential Geometry. In fact, I have taught such a course for mathematics, physics, engineering, chemistry, biology and philosophy majors and I have used topics from Chapters 1, 2, 3, 4, 5, 6 and 7. This does not mean that I have covered these chapters exhaustively in one quarter, but that I have chosen certain parts to emphasize and allowed students to do projects, say, involving other parts. For example, students have done projects on involutes and gear teeth design, re-creation of curves from curvature and torsion, Enneper's surface and area minimization (see above), geodesics on minimal surfaces and the Euler-Lagrange equations in relativity. In many cases, students have gone way beyond what this book contains and I owe them my thanks for expanding my knowledge. The book, as it now stands, is suitable for either a one-quarter or one-semester course in Differential Geometry as well as a full-year course. In the case of the latter, all Chapters may be completed. In the case of the former, I would recommend the chapters I've listed above, but there is a good choice of alternative material as well.

The reader should note two things about the layout of the book. First, the exercises are integrated into the text. While this may make them somewhat harder to find, it also makes them an essential part of the text. The reader should at least *read* the exercises when going through a chapter — they are important. Secondly, I have chosen to number theorems, lemmas, examples, definitions and remarks in order as is usually done using LaTeX.

There are several students who deserve special mention with regard to this text. Rob Clark first interested me in minimal surfaces and, together with Jack Chen, showed me the use of computers (e.g. Ken Brakke's Evolver program) in distinguishing 'minimal' from 'harmonic'. Laszlo Ilyes provided many of the Maple procedures for optimal control while Carrie Kyser took my original laughable "geodesic procedures" and transformed them wonderfully into one elegant procedure which does exactly what I want it to do. Sue Halamek did an excellent job on the first draft of the solutions to problems and any present errors are certainly due to my final editing. Thanks to you all and to all the students who watched me fumble my way to a book!

I would also like to acknowledge the contributions of my friend Allen Broughton. It was Allen who first taught a course from a sheaf of my handwritten notes and actually made sense of the notes and a success of the course. Allen also first explored the use of Maple for Differential Geometry and is responsible for producing the first procedures for calculating curvatures etc. Similarly, the handwritten notes referred to above would have remained just that without the TEX-pertise of Joyce Pluth. Joyce typed the first draft of those notes and patiently tutored me in the intricacies of TEXuntil I stopped bothering her. Let me also thank Elaine Hoff and Dena Jones for helping me to photocopy, collate, cut and paste to ready versions of this text for unsuspecting classes. I must also thank the members of the Cleveland Geometry/Topology Seminar for sitting through numerous lectures on various parts of this text.

Finally, the writing of this book would have been impossible without the help, advice and understanding of my wife Jan and daughter Kathy. Thanks — the computer is now free!

John Oprea
Cleveland, Ohio
oprea@math.csuohio.edu

Note: See the website *www.csuohio.edu/math/oprea* for Maple updates.

Addendum to Second Edition

Since the publication of the First Edition, many people have sent me comments, suggestions, and corrections. I have tried to take all of these into account in preparing the Second Edition, but sometimes this has proved to be impossible. One reason for this is that I want to keep the book at a level that is truly accessible to undergraduates. So, for me, some arguments simply can't be made. On the other hand, I have learned a great deal from all of the comments sent to me and, in some sense, this is the real payment for writing the book. Therefore, I want to acknowledge a few people who went beyond the call of duty to give me often extensive commentary. These folks are (in alphabetical order!): David Arnold, David Bao, Neil Bomberger, Gary Crum, Dan Drucker, Lisbeth Fajstrup, Karsten Grosse-Brauckmann, Sigmundur Gudmundsson, Greg Lupton, Takashi Kimura, Jaak Peetre, Ted Shifrin, and Peter Stiller. Thanks to all of you.

The Second Edition, of course, contains corrections to misprints and mathematical errors which found their way into the First Edition, but it also contains new material. In particular, in recent years I have become convinced of the utility of the elliptic functions in differential geometry and the calculus of variations, so I have included a simplified, straightforward introduction to these here. The main applications of elliptic functions presented here are the derivation of explicit parametrizations for unduloids and for the Mylar balloon. Such explicit parametrizations allow for the determination of differential geometric invariants such as Gauss curvature as well as an analysis of geodesics. Of course, part of this analysis involves Maple. These applications of elliptic functions are distillations of joint work with Ivailo Mladenov, and I want to acknowledge that here with thanks to him for his insights and diligence concerning this work.

The Maple work found in the Second Edition once again focuses on actually doing interesting things with computers rather than simply drawing pictures. Nevertheless, in transporting the book from the AMS-TeX of the First Edition to the LaTeX2e of the Second, it has proved to be much easier to embed encapsulated Postscript files. So there are many more pictures of interesting phenomena in this edition. The pictures have all been created by me with Maple. In fact, by examining the Maple sections at the ends of chapters, it is usually pretty clear how all pictures were created. The version of Maple used for this edition is Maple 8. The Maple work in the First Edition needed extensive revision to work with Maple 8 because Maple developers changed the way certain commands work. I have been personally assured by these developers that this will not happen in the future — we will see. Should newer versions of Maple cause problems for the procedures in this book, look at my website listed for updates: **www.csuohio.edu/math/oprea**. One thing to pay attention to concerning this issue of Maple command changes is the following. Maple no longer supports the "linalg" package. Rather, Maple has moved to a package called "LinearAlgebra" and I have changed all Maple work in the book to reflect this. This should be stable for some time to come, no matter what new versions of Maple arise. Of course, the one thing that doesn't change is the book's focus on the solutions of differential equations as the heart of differential geometry. Because of this, Maple plays an even more important role through its "dsolve" command and its ability to solve differential equations explicitly and numerically.

Originally this book was intended for a one-quarter or one-semester course in the geometry of curves and surfaces. Now, however, it seems to have grown beyond this, so I would like to make some recommendations for instructors who do not already have their syllabi set in stone. A good one-semester course can be obtained from Chapter 1, Chapter 2, Chapter 3, and the first "half" of Chapter 5. This carries students through the basic geometry of curves and surfaces while introducing various curvatures and applying virtually all of these ideas to study geodesics. My personal predilections would lead me to use Maple extensively to foster a certain geometric intuition. I also might use material such as the industrial application of Section 5.7 as a student group project for the semester. A second semester course could focus on the remainder of Chapter 5, Chapter 6, and Chapter 7 while saving Chapter 4 on minimal surfaces or Chapter 8 on higher dimensional geometry for projects. Students then will have seen Gauss-Bonnet, holonomy, and a kind of recapitulation of geometry (together with a touch of mechanics) in terms of the Calculus of Variations. There are, of course, many alternative courses hidden within the book and I can only wish "good hunting" to all who search for them.

John Oprea
Cleveland, Ohio
oprea@math.csuohio.edu
j.oprea@csuohio.edu
www.csuohio.edu/math/oprea

Note: Maple 9 appeared in Summer of 2003 and everything in the book has been tested with it. All commands and procedures work *with one small exception.* On page 167 in Chapter 3, the following Maple code appears to define a surface of revolution with functions g and h.

```
> h:=t->h(t);g:=t->g(t);
```

$$h := h$$

$$g := g$$

```
> surfrev:=[h(u)*cos(v),h(u)*sin(v),g(u)];
```

$$surfrev := [\mathrm{h}(u)\cos(v),\ \mathrm{h}(u)\sin(v),\ \mathrm{g}(u)]$$

This works fine in Maple 8, but Maple 9 complains about defining g and h this way saying that there are too many levels of recursion in the formula for "surfrev". The fix for Maple 9 is simple. *Just don't define* g and h at all! Go straight to the definition of "surfrev". Then everything else works. The same type of definition trouble occurs on pages 214, 215, and 217 and the fix is the same.

Note to Students

Every student who takes a mathematics class wants to know what the real point of the course is. Often, courses proceed by going through a list of topics with accompanying results and proofs and, while the rationale for the ordering and presentation of topics may be apparent to the instructor, this is far from true for students. Books are really no different; authors get caught up in the "material" because they love their subject and want to show it off to students. So let's take a moment now to say what the point of differential geometry is from the perspective of this text.

Differential geometry is concerned with understanding shapes and their properties in terms of calculus. We do this in two main ways. We start by defining shapes using "formulas" called parametrizations and then we take derivatives and algebraically manipulate them to obtain new expressions that we show represent actual geometric entities. So, if we have geometry encoded in the algebra of parametrizations, then we can derive quantities telling us something about that geometry from calculus. The prime examples are the various curvatures which will be encountered in the book. Once we see how these special quantities arise from calculus, we can begin to turn the problem around by restricting the quantities in certain ways and asking what shapes have quantities satisfying these restrictions. For instance, once we know what curvature means, we can ask what plane curves have curvature functions that are constant functions. Since this is, in a sense, the reverse of simply calculating geometric quantities by differentiation, we should expect that "integration" arises here. More precisely, conditions we place on the geometric quantities give birth to differential equations whose solution sets "are" the shapes we are looking for.

So differential geometry is intimately tied up with differential equations. But don't get the idea that all of those crazy methods in a typical differential equations text are necessary to do basic differential geometry. Being able to handle separable differential equations and knowing a few tricks (which can be learned along the way) are usually sufficient. Even in cases where explicit solutions to the relevant differential equations don't exist, numerical solutions can often produce a solution shape. The advent of computer algebra systems in the last decade makes this feasible even for non-experts in computer programming.

In Chapter 1, we will treat the basic building blocks of all geometry, curves, and we will do exactly as we have suggested above. We will use calculus to develop a system of differential equations called the Frenet equations that determine a curve in three-dimensional space. We will use the computer algebra system Maple to numerically solve these equations and plot curves in space. But with any computer program there are inputs, and these are the parameters involved in the Frenet equations; the curvature and torsion of a curve. So, we are saying that the curvature and torsion determine a curve in a well-defined sense. This is exactly the program outlined above. Of course, we will also see that, by putting restrictions on curvature and torsion, we can see what curves arise analytically as well.

In Chapter 2, we take what we have learned about curves and apply it to study the geometry of surfaces in 3-space. The key definition is that of the shape operator because from it flows all of the rest of the types of curvatures we use to understand geometry. The shape operator is really just a way take derivatives "in a tangential direction" and is related to the usual directional derivative in 3-space. What is interesting here is that the shape operator can be thought of as a matrix and this allows us to actually define curvatures in terms of the linear algebraic invariants of the matrix. For instance, principal curvatures are simply the eigenvalues of the shape operator, mean curvature is the average of the eigenvalues (i.e. one half the trace of the matrix) and Gauss curvature is the product of the eigenvalues (i.e. the determinant of the matrix). Of course, the challenge now is twofold: first, show that the shape operator and its curvature offspring reflect our intuitive grasp of the geometry of surfaces and, secondly, show that these curvatures are actually computable. This leads to the next chapter.

In Chapter 3, we show that curvatures are computable just in terms of derivatives of a parametrization. This not only makes curvatures computable, but allows us to put certain restrictions on curvatures and produce analytic solutions. For instance, we can really see what surfaces arise when Gauss curvature is required to be constant on a compact surface or when mean curvature is required to be zero on a surface of revolution. An important byproduct of this quest for computability is a famous result of Gauss that shows that Gauss curvature can be calculated directly from the metric; that is, the functions which tell us how the surface distorts usual Euclidean distances. The reason this is important is that it gives us a definition of curvature that can be transported out of 3-space into a more abstract world of surfaces. This is the first step towards a more advanced differential geometry.

Chapter 4 deals with minimal surfaces. These are surfaces with mean curvature equal to zero at each point. Our main theme shows up here when we show that minimal surfaces (locally) satisfy a (partial) differential equation known as the minimal surface equation. Moreover, by putting appropriate restrictions on the surface's defining function, we will see that it is possible to solve the minimal surface equation analytically. From a more geometric (as opposed to analytic) viewpoint, we focus here on basic computations and results, as well as the interpretations of soap films as minimal surfaces and soap bubbles as surfaces where the mean curvature is a constant function. The most important result along these lines is Alexandrov's theorem, where it is shown that such a compact surface embedded in 3-space must be a sphere. The chapter also discusses harmonic functions and this leads to a more advanced approach to minimal surfaces, but from a more analytic point of view. In particular, we introduce complex variables as the natural parameters for a minimal surface. We don't expect the reader to have any experience with complex variables (beyond knowing what a complex number is, say), so we review the relevant aspects of the subject. This approach produces a wealth of information about minimal surfaces, including an example where a minimal surface does not minimize surface area.

In Chapter 5, we start to look at what different geometries actually tell us. A fundamental quality of a "geometry" is the type of path which gives the shortest distance between points. For instance, in the plane, the shortest distance between points is a straight line, but on a sphere this is no longer the case. If we go from Cleveland to Paris, unless we are very good at tunnelling, we must take the great circle route to achieve distance minimization. Knowing that shortest length curves are great circles on a sphere gives us an intuitive understanding of the curvature and symmetry of the sphere. So this chapter deals with "shortest length curves" (i.e. geodesics) on a surface. In fact, we modify this a bit to derive certain differential equations called the geodesic equations whose solutions are geodesics on the surface. Again, while it is sometimes possible to obtain analytic expressions for geodesics, more often we solve the geodesic equations numerically and plot geodesics to discover the underlying geometry of the surfaces. The geodesic equations may also be transported to a more abstract situation, so we begin to see more general geometric effects here as well.

Chapter 6 is the culmination of much of what has come before. For in this chapter, we see how curvature can affect even the most basic of geometric qualities, the sum of the angles in a triangle. The formalization of this effect, which is one of the most beautiful results in Mathematics, is known as the Gauss-Bonnet theorem. We present various applications of this theorem to show how "abstract" results can produce concrete geometric information. Also in this chapter, we introduce a notion known as holonomy that has profound effects in physics, ranging from classical to quantum mechanics. In particular, we present holonomy's effect on the precession of Foucault's pendulum, once again demonstrating the influence of curvature on the world in which we live.

Chapter 7 presents what can fairly be said to be the prime philosophical underpinning of the relationship of geometry to Nature, the calculus of variations. Physical systems often take a configuration determined by the minimization of potential energy. For instance, a hanging rope takes the shape of a catenary for this reason. Generalizing this idea leads yet again to a differential equation, the Euler-Lagrange equation, whose solutions are candidates for minimizers of various functionals. In particular, Hamilton's principle says that the motions of physical systems arise as solutions of the Euler-Lagrange equation associated to what is called the action integral. A special case of this gives geodesics and we once again see geometry arising from a differential equation (which itself is the reflection of a physical principle).

In Chapter 8, we revisit virtually all of the earlier topics in the book, but from the viewpoint of manifolds, the higher-dimensional version of surfaces. This is necessarily a more abstract chapter because we cannot see beyond three dimensions, but for students who want to study physics or differential geometry, it is the stepping stone to more advanced work. Systems in Nature rarely depend on only two parameters, so understanding the geometry inherent in larger parameter phenomena is essential for their analysis. So in this chapter, we deal with minimal submanifolds, higher-dimensional geodesic equations and the Riemann, sectional, Ricci and scalar curvatures. Since these topics are the subjects of many volumes themselves, here we only hope to indicate their relation to the surface theory presented in the first seven chapters.

So this is the book. The best advice for a student reading it is simply this: look for the right differential equations and then try to solve them, analytically or numerically, to discover the underlying geometry. Now let's begin.

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