

Topics in Contemporary Differential Geometry, Complex Analysis and Mathematical Physics

Proceedings of the 8th International Workshop on Complex Structures and Vector Fields

Stancho Dimiev
Kouei Sekigawa

Editors

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PREFACE

This book is the Proceedings of the 8th International Workshop on Complex Structures and Vector Fields held at Bulgarian Academy of Sciences, Institute of Mathematics and Informatics (Sofia) from August 21 to August 2, 2006. The first Workshop was held at the same place on 1992. We are aiming at the higher achievement of the studies of current topics in Complex Analysis, Differential Geometry, Mathematical Physics and also of the intermediate ones among them including their applications. It is notable that many new specialists in Mathematical Physics attended the present Workshop besides regular participants in the previous Workshops and also that a new tendency to expand our subject matters is adopted in the present Workshop, and places especially emphases on the further development of the studies in Differential Geometry, Complex Analysis, Partial Differential Equations and Integrable System, and also on the expansion of the research areas including new ones in Mathematical Physics in the forthcoming Workshops.

This book is dedicated to the memory of three distinguished scientists, Professor Shigeru Ishihara who is regarded as a teacher of many Japanese participants, Professor Shozo Koshi who was an active participant of the Workshop, and Professor Sawa Manoff who was an active participant of the Workshop and also made much effort for the development of the same Workshop.

The editors express their deepest gratitude to Professor T. Oguro for his outstanding co-operation and efforts in the arrangements of this volume.

Editors

THE 8TH INTERNATIONAL WORKSHOP ON COMPLEX STRUCTURES AND VECTOR FIELDS

21-26 August 2006, Sofia – Bulgaria

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1. Yasuo Matsushita
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2. Peter Popivanov
On the hypoellipticity of complex valued vector fields
3. Toshiaki Adachi
Killing helices on a complex projective space and canonical magnetic fields on geodesic spheres
4. Vestislav Apostolov
Generalized Kähler manifolds, commuting complex structures and split tangent bundles
5. Rossen Dandoloff
Anholonomy of a moving space curve and the Schrödinger equation
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On almost Hermitian structures of 6-dimensional submanifolds in the octonions
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Nonexistence of nonzero resonances for Schrödinger operators with singular perturbation
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MODULI SPACE OF KILLING HELICES OF LOW ORDERS ON A COMPLEX SPACE FORM

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We give a report on the moduli space of helices of proper order less than 5 which are generated by some Killing vector fields on a complex space form from the viewpoint of the length spectrum.

1. Introduction

In this note we give a summary of my work concerning essential Killing helices of low orders on a non-flat complex space form, which is either a complex projective space or a complex hyperbolic space. A smooth curve γ parameterized by its arclength on a Riemannian manifold M is said to be a *helix of proper order d* if it satisfies the following system of ordinary differential equations

$$\nabla_{\dot{\gamma}} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1}, \quad 1 \leq j \leq d, \quad (1.1)$$

with positive constants $\kappa_1, \dots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot{\gamma}, Y_2, \dots, Y_d\}$ of vector fields along γ . Here $\kappa_0 = \kappa_d = 0$, and Y_0, Y_{d+1} are null vector fields along γ . These constants $\kappa_1, \dots, \kappa_{d-1}$ are called the *geodesic curvatures* of γ and the system $\{Y_i\}$ the *Frenet frame* of γ . We call a helix *Killing* if it is generated by some Killing vector field on M . On real space forms, which are standard spheres, Euclidean spaces and real hyperbolic spaces, all helices are Killing and lengths of closed helices are given by their geodesic curvatures. But on a complex space form the situation is different. We study the difference on laminations on the moduli spaces of Killing helices which are induced by the length spectrum.

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2. Moduli space of Killing helices and length spectrum

We say two helices γ_1, γ_2 on a Riemannian manifold M are *congruent* to each other if there exist an isometry φ of M and a constant t_0 satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for all t . We denote by $\mathcal{K}_d(M)$ the set of all congruence classes of Killing helices of proper order d on M . We put $\mathcal{K}(M) = \bigcup_{d=1}^{\infty} \mathcal{K}_d(M)$ and call it the *moduli space* of Killing helices. On a real space form $\mathbb{R}M^n$, as helices are classified by their geodesic curvatures, we see $\mathcal{K}_d(\mathbb{R}M^n)$ is bijective to $(0, \infty)^{d-1}$, the $(d-1)$ product of half lines, when $d \leq n$. But on a non-flat complex space form, as isometries are either holomorphic or anti-holomorphic, the moduli space of Killing helices is not so simple. For a helix on a Kähler manifold (M, J) with Frenet frame $\{Y_i\}$, we define its *complex torsions* τ_{ij} ($1 \leq i < j \leq d$) by $\tau_{ij} = \langle Y_i, JY_j \rangle$. As was pointed out in [8], on a non-flat complex space form $\mathbb{C}M^n$ a helix γ is Killing if and only if all its complex torsions are constant along γ .

We call a helix γ *closed* if there is positive t_c with $\gamma(t + t_c) = \gamma(t)$ for all t . The minimum positive t_c with this property is called the length of γ and is denoted by $\text{length}(\gamma)$. When γ is not closed we say it is *open* and put $\text{length}(\gamma) = \infty$. The *length spectrum* $\mathcal{L} : \mathcal{K}(M) \rightarrow (0, \infty]$ is defined by $\mathcal{L}([\gamma]) = \text{length}(\gamma)$, where $[\gamma]$ denotes the congruence class containing a helix γ . For the sake of simplicity we denote a restriction of \mathcal{L} onto a subset of $\mathcal{K}(M)$ also by \mathcal{L} .

3. Moduli space of helices on a real space form

For the sake of comparison, we here show some properties on length spectrum of helices on a real space form $\mathbb{R}M^n(c)$ of constant sectional curvature c . The length spectrum $\mathcal{L} : \mathcal{K}_2(\mathbb{R}M^n(c)) \cong (0, \infty) \rightarrow (0, \infty]$ of circles of positive geodesic curvature, which are helices of proper order 2, is given as $\mathcal{L}(\kappa) = 2\pi/\sqrt{\kappa^2 + c}$, where we read it infinity when $\kappa^2 + c \leq 0$. Thus if we induce the canonical Euclidean differential structure on $\mathcal{K}_2(\mathbb{R}M^n(c))$, we see the length spectrum is smooth on this moduli space.

For about the moduli space of helices of proper order 3 on $\mathbb{R}M^n(c)$ ($n \geq 3$), the feature depends on sectional curvature c . All helices of proper order 3 on a Euclidean space \mathbb{R}^n are unbounded. For a standard sphere $S^n(c)$ of constant sectional curvature c , we have a canonical foliation $\{\mathcal{G}_\alpha\}_{\alpha \in (1, \infty)}$ on $\mathcal{K}_3(S^n(c))$ which is related with the length spectrum and is given as

$$\mathcal{G}_\alpha = \{ [\gamma_{\kappa_1, \kappa_2}] \mid \kappa_1^2 + (\kappa_2 - \alpha\sqrt{c}/2)^2 = c(\alpha^2 - 1)/4 \},$$

where $[\gamma_{\kappa_1, \kappa_2}]$ denotes the congruence class of helices of proper order 3 on $S^n(c)$ with geodesic curvatures κ_1, κ_2 (see Figure 1).

Theorem 3.1. *The length spectrum $\mathcal{L} : \mathcal{K}_3(S^n(c)) \rightarrow (0, \infty]$ is constant on each leaf. Each leaf is set theoretically maximal with respect to this property. A leaf \mathcal{G}_α consists of congruence classes of closed helices if and only if α and $\sqrt{\alpha^2 - 1}$ are rational.*

For a real hyperbolic space $H^n(c)$ of constant sectional curvature c , we have a canonical foliation $\{\mathcal{G}_\alpha\}_{\alpha \in (-\infty, \infty)}$ on $\mathcal{K}_3(H^n(c))$ which is given as

$$\mathcal{G}_\alpha = \{ [\gamma_{\kappa_1, \kappa_2}] \mid \kappa_1^2 + (\kappa_2 - \alpha\sqrt{|c|}/2)^2 = -c(\alpha^2 + 1)/4 \}.$$

We should note that the moduli space $\mathcal{BK}_3(H^n(c))$ of bounded helices of proper order 3 on $H^n(c)$ is given as $\{ [\gamma_{\kappa_1, \kappa_2}] \mid \kappa_1^2 + (\kappa_2 - \sqrt{|c|}/2)^2 > -c/2 \}$. On this space the foliation $\{\mathcal{G}_\alpha\}_{\alpha \in (1, \infty)}$ satisfies the same property as of the foliation on $\mathcal{K}_3(S^n)$. In both cases of a standard sphere and of a real hyperbolic space, these foliations can be naturally extend to a foliation or a lamination on $\mathcal{K}_2(\mathbb{R}M^n(c)) \cup \mathcal{K}_3(\mathbb{R}M^n(c))$.

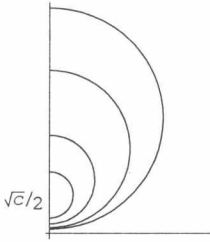


Fig. 1. Foliation on $\mathcal{K}_3(S^n(c))$

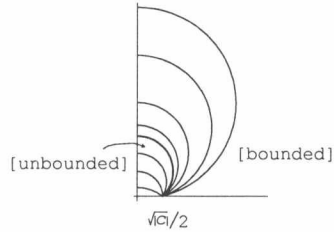


Fig. 2. Foliation on $\mathcal{K}_3(H^n(c))$

4. Moduli space of circles on a complex space form

We now study the moduli space of helices on a non-flat complex space form. On a Kähler manifold, the complex torsion τ_{12} of each circle γ is always constant along γ , because

$$\tau'_{12} = \langle \nabla_{\dot{\gamma}} \dot{\gamma}, JY_2 \rangle + \langle \dot{\gamma}, J\nabla_{\dot{\gamma}} Y_2 \rangle = \kappa_1 (\langle Y_2, JY_2 \rangle - \langle \dot{\gamma}, J\dot{\gamma} \rangle) = 0.$$

Therefore we see the moduli space $\mathcal{K}_2(\mathbb{C}M^n)$ of circles of positive geodesic curvature on a non-flat complex space form is set theoretically bijective to the product $(0, \infty) \times [0, 1]$ when $n \geq 2$. In this section we suppose $n \geq 2$ and we shall denote by $[\gamma_{\kappa, \tau}]$ the congruence class of circles with geodesic curvature κ and complex torsion $\tau_{12} = \tau$ on a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c .

For a complex projective space $\mathbb{C}P^n(c)$, we have a lamination structure $\{\mathcal{F}_\mu\}_{\mu \in [0,1] \cup \{\star\}}$ on $\mathcal{K}_2(\mathbb{C}P^n(c))$ defined by

$$\mathcal{F}_\mu = \begin{cases} \{[\gamma_{\kappa,0}] \mid \kappa > 0\}, & \text{if } \mu = 0, \\ \{[\gamma_{\kappa,\tau}] \mid 3\sqrt{3}c\kappa\tau(4\kappa^2 + c)^{-3/2} = \mu\}, & \text{if } 0 < \mu < 1, \\ \{[\gamma_{\kappa,1}] \mid \kappa > 0\}, & \text{if } \mu = \star. \end{cases}$$

Theorem 4.1. *The length spectrum $\mathcal{L} : \mathcal{K}_2(\mathbb{C}P^n(c)) \rightarrow (0, \infty]$ is smooth on each leaf with respect to the canonical induced Euclidean differential structure. Each leaf is maximal with respect to this property.*

- 1) The leaf \mathcal{F}_\star consists of congruence classes of closed circles satisfying $\mathcal{L}([\gamma_{\kappa,1}]) = 2\pi/\sqrt{\kappa^2 + c}$.
- 2) The leaf \mathcal{F}_0 also consists of congruence classes of closed circles satisfying $\mathcal{L}([\gamma_{\kappa,0}]) = 4\pi/\sqrt{4\kappa^2 + c}$.
- 3) The leaf \mathcal{F}_μ ($0 < \mu < 1$) consists of congruence classes of closed circles if and only if $\mu = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with some relatively prime positive integers p, q satisfying $p > q$. On this leaf $\mathcal{L}([\gamma_{\kappa,\tau}]) = 2\delta(p, q)\pi\sqrt{(3p^2 + q^2)/\{3(4\kappa^2 + c)\}}$, where $\delta(p, q) = 1$ when the product pq is odd and $\delta(p, q) = 2$ when pq is even.

For a complex hyperbolic space $\mathbb{C}H^n(c)$, we also have a lamination structure $\{\mathcal{F}_\mu\}_{\mu \in [0,\infty] \cup \{\star\}}$ on $\mathcal{K}_2(\mathbb{C}H^n(c))$ defined by

$$\mathcal{F}_\mu = \begin{cases} \{[\gamma_{\kappa,0}] \mid \kappa > 0\}, & \text{if } \mu = 0, \\ \{[\gamma_{\kappa,\tau}] \mid 3\sqrt{3}|c|\kappa\tau|4\kappa^2 + c|^{-3/2} = \mu\}, & \text{if } 0 < \mu < \infty, \\ \{[\gamma_{\sqrt{|c|/2},\tau}] \mid 0 < \tau < 1\}, & \text{if } \mu = \infty, \\ \{[\gamma_{\kappa,1}] \mid \kappa > 0\}, & \text{if } \mu = \star. \end{cases}$$

This lamination has the same properties as of the lamination on $\mathcal{K}_2(\mathbb{C}P^n)$ if we restrict ourselves on the moduli space

$$B\mathcal{K}_2(\mathbb{C}H^n(c)) = \{[\gamma_{\kappa,\tau}] \mid 0 \leq \tau < \nu(\kappa)\} \cup \{[\gamma_{\kappa,1}] \mid \kappa > \sqrt{|c|}\}$$

of bounded circles on $\mathbb{C}H^n(c)$ (see [2]). Here $\nu : (0, \infty) \rightarrow [0, 1]$ is given by

$$\nu(\kappa) = \begin{cases} 0, & \text{if } 0 < \kappa \leq \sqrt{|c|}/2, \\ (4\kappa^2 + c)^{3/2}/(3\sqrt{3}|c|\kappa), & \text{if } \sqrt{|c|}/2 < \kappa < \sqrt{|c|}, \\ 1, & \text{if } \kappa \geq \sqrt{|c|}. \end{cases}$$

In view of the features of these laminations on the moduli spaces of circles, we find the set $\{[\gamma_{\kappa,1}] \mid \kappa > 0\}$ of congruence classes of trajectories

for Kähler magnetic fields is quite different from other part of $\mathcal{K}_2(\mathbb{C}M^n(c))$. Since each trajectory lies on some totally geodesic $\mathbb{C}M^1$ and other circles do not lie on $\mathbb{C}M^1$, we shall classify helices by this property. We call a helix on $\mathbb{C}M^n$ of proper order $2k - 1$ or $2k$ essential if it lies on some totally geodesic $\mathbb{C}M^k$. We denote by $\mathcal{EK}_d(\mathbb{C}M^n(c))$ the set of all congruence classes of essential Killing helices of proper order d on $\mathbb{C}M^n(c)$.



Fig. 3. Lamination on $\mathcal{K}_2(\mathbb{C}P^n(c))$

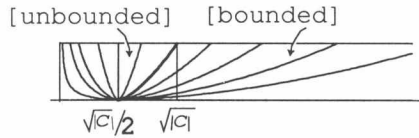


Fig. 4. Lamination on $\mathcal{K}_2(\mathbb{C}H^n(c))$

5. Moduli spaces of Killing helices of orders less than 5 on a complex space form

Though all circles on a non-flat complex space form are Killing, helices of proper order greater than 2 are not necessarily Killing. Computing τ'_{ij} by using (1.1) we see a helix of proper order d on $\mathbb{C}M^n$ is Killing if and only if

$$-\kappa_{i-1}\tau_{i-1j} + \kappa_i\tau_{i+1j} - \kappa_{j-1}\tau_{ij-1} + \kappa_j\tau_{ij+1} = 0, \quad 1 \leq i < j \leq d, \quad (5.1)$$

where we set $\tau_{0k} = \tau_{kk} = \tau_{kd+1} = 0$ ([7]). Applying these relations to a helix of proper order 3 on $\mathbb{C}M^n$ ($n \geq 2$), we find it is Killing if and only if its geodesic curvatures and complex torsions satisfy $\tau_{13} = 0$ and $\kappa_1\tau_{23} = \kappa_2\tau_{12}$. If we consider the initial frame we find the following.

- 1) A helix is essential Killing if and only if $\tau_{12} = \pm\kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$, $\tau_{13} = 0$, $\tau_{23} = \pm\kappa_2/\sqrt{\kappa_1^2 + \kappa_2^2}$, where the double signs take the same signature.
- 2) When $n \geq 3$, a helix is Killing if and only if its complex torsions satisfy $\tau_{12} = \kappa_1\tau$, $\tau_{13} = 0$, $\tau_{23} = \kappa_2\tau$ with some τ satisfying $|\tau| \leq 1/\sqrt{\kappa_1^2 + \kappa_2^2}$.

Thus we see the moduli space $\mathcal{EK}_3(\mathbb{C}M^n)$ of essential Killing helices of proper order 3 is bijective to a quarter of a plane $(0, \infty)^2$ and the moduli space $\mathcal{K}_3(\mathbb{C}M^n)$ is bijective to the set $(0, \infty)^2 \times [0, 1]$.

When we consider Killing helices of proper order 4, the relations (5.1) turn to $\kappa_1\tau_{23} + \kappa_3\tau_{14} = \kappa_2\tau_{12}$, $\kappa_3\tau_{23} + \kappa_1\tau_{14} = \kappa_2\tau_{34}$ and $\tau_{13} = \tau_{24} = 0$. Considering the initial frame we find a helix of proper order 4 on $\mathbb{C}M^n$ ($n \geq 2$) is essential Killing if and only if its complex torsions satisfy one of the following conditions: