Banesh Hoffmann

ABOUT VECTORS

BANESH HOFFMANN

Professor of Mathematics
Queens College of the City University of New York

ABOUT VECTORS

DOVER PUBLICATIONS, INC., NEW YORK

Books by Banesh Hoffmann

The Strange Story of the Quantum
The Tyranny of Testing
About Vectors
Albert Einstein, Creator and Rebel
(with Helen Dukas)

Published in Canada by General Publishing Company, Ltd., 30 Lesmill Road, Don Mills, Toronto, Ontario.
Published in the United Kingdom by Constable and Company, Ltd.

This Dover edition, first published in 1975, is an unabridged, corrected republication of the work originally published by Prentice-Hall, Inc., New Jersey, in 1966.

International Standard Book Number: 0-486-60489-6
Library of Congress Catalog Card Number: 75-11081

Manufactured in the United States of America
Dover Publications, Inc.
180 Varick Street
New York, N.Y. 10014

As far as the laws of mathematics
refer to reality, they are not certain;
and as far as they are certain,
they do not refer to reality.

ALBERT EINSTEIN

PREFACE

This book is written as much to disturb and annoy as to instruct. Indeed, it seeks to instruct primarily by being disturbing and annoying, and it is often deliberately provocative. If it should cause heated discussion and a re-examination of fundamentals in classroom and mathematics club it will have achieved one of its main purposes.

It is intended as a supplement and corrective to textbooks, and as collateral reading in all courses that deal with vectors. Because the exercises call for no great manipulative skill, and the book avoids using the calculus, it may at first sight seem to be elementary. But it is not. It has something for the beginner, to be sure. But it also has something for quite advanced students—and something, too, for their instructors.

I have tried to face awkward questions rather than achieve a spurious simplicity by sweeping them under the rug. To counteract the impression that axioms and definitions are easily come by and that mathematics is a thing of frozen beauty rather than something imperfect and growing, I have mixed pure and applied mathematics and have made the problem of defining vectors a developing, unresolved *leitmotif*. The book is unconventional, and to describe it further here would be to blunt its intended effect by giving away too much of the plot. A brief word of warning will not be amiss, however. There are no pat answers in this book. I often present ideas in conventional form only to show later that they need modification because of unexpected difficulties, my aim being to induce a healthy skepticism. But too much healthy skepticism can be decidedly unhealthy. The reader should therefore realize that the ideas could have been presented far more hearteningly as a sequence of ever-deepening insights and, thus, of successive mathematical triumphs rather than defeats. If he reads between the lines he will see that, in a significant sense, they are indeed so presented.

To my friends Professors Arthur B. Brown and Václav Hlavatý, who read the manuscript, go my warmest thanks. It is impossible to express the depth of my indebtedness to them for their penetrating comments, which have led to major improvements in the text. They should not be held accountable for the views expressed in the book: on some issues I resisted the urgent advice of one or the other of them. A ground-breaking book of this sort is unlikely to be free of debatable views and outright errors, and for all of these I bear the sole responsibility.

BANESH HOFFMANN

CONTENTS

1

INTRODUCING VECTORS

1. DEFINING A VECTOR

Making good definitions is not easy. The story goes that when the philosopher Plato defined *Man* as "a two-legged animal without feathers," Diogenes produced a plucked cock and said "Here is Plato's man." Because of this, the definition was patched up by adding the phrase "and having broad nails"; and there, unfortunately, the story ends. But what if Diogenes had countered by presenting Plato with the feathers he had plucked?

Exercise 1.1 What? [Note that Plato would now have feathers.]

Exercise 1.2 Under what circumstances could an elephant qualify as a man according to the above definition?

A *vector* is often defined as *an entity having both magnitude and direction.* But that is not a good definition. For example, an arrow-headed line segment like this

has both magnitude (its length) and direction, and it is often used as a drawing of a vector; yet it is not a vector. Nor is an archer's arrow a vector, though it, too, has both magnitude and direction.

To define a vector we have to add to the above definition something analogous to "and having broad nails," and even then we shall find ourselves not wholly satisfied with the definition. But it will let us start, and we can try patching up the definition further as we proceed—and we may even find ourselves replacing it by a quite different sort of definition later on. If, in the end, we have the uneasy feeling that we have still not found a completely satisfactory definition of a vector, we need not be dismayed, for it is the nature of definitions not to be completely satisfactory, and we shall have learned pretty well what a vector is anyway, just as we know, without being able to give a satisfactory definition, what a man is—well enough to be able to criticize Plato's definition.

Exercise 1.3 Define a *door*.

Exercise 1.4 Pick holes in your definition of a *door*.

Exercise 1.5 According to your definition, is a movable partition between two rooms a door?

2. THE PARALLELOGRAM LAW

The main thing we have to add to the magnitude-and-direction definition of a vector is the following:

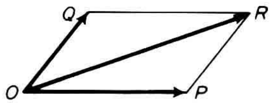


Figure 2.1

Let us think of vectors as having definite locations. And let the arrow-headed line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in Figure 2.1 represent the magnitudes, directions, and locations of two vectors starting at a common point O. Complete the parallelogram formed by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, and draw the diagonal OR. Then, when taken together, the two vectors represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are equivalent to a single vector represented by the arrow-headed line segment $\overrightarrow{OR}$. This vector is called the *resultant* of the vectors represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, and the above crucial property of vectors is called the *parallelogram law* of combination of vectors.

Exercise 2.1 Find (*a*) by drawing and measurement, and (*b*) by calculation using Pythagoras' theorem, the magnitude and direction of the resultant of two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ if each has magnitude 3, and $\overrightarrow{OP}$ points thus $\longrightarrow$ while $\overrightarrow{OQ}$ points perpendicularly, thus $\uparrow$. [*Ans.* The magnitude is $3\sqrt{2}$, or approximately 4.2, and the direction bisects the right angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.]

Exercise 2.2 Show that the resultant of two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ that point in the same direction is a vector pointing in the same direction and having a magnitude equal to the sum of the magnitudes of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. [Imagine the parallelogram in Figure 2.1 squashed flat into a line.]

Exercise 2.3 Taking a hint from Exercise 2.2, describe the resultant of two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ that point in opposite directions.

Exercise 2.4 In Exercise 2.3, what would be the resultant if $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ had equal magnitudes? [Do you notice anything queer when you compare this resultant vector with the definition of a vector?]

Exercise 2.5 Observe that the resultant of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is the same as the resultant of $\overrightarrow{OQ}$ and $\overrightarrow{OP}$. [This is trivially obvious, but keep it in mind nevertheless. We shall return to it later.]

In practice, all we need to draw is half the parallelogram in Figure 2.1—either triangle OPR or triangle OQR. When we do this it looks as if we had combined two vectors $\overrightarrow{OP}$ and $\overrightarrow{PR}$ (or $\overrightarrow{OQ}$ and $\overrightarrow{QR}$) end-to-end like this, even

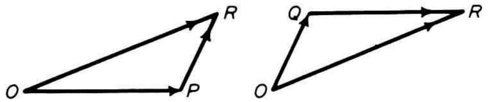
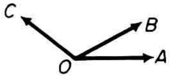


Figure 2.2 (For clarity, the arrow heads meeting at R have been slightly displaced. We shall occasionally displace other arrow heads under similar circumstances.)

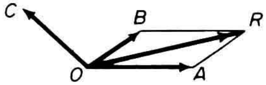
though they do not have the same starting point. Actually, though, we have merely combined $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ by the parallelogram law.* But suppose we were dealing with what are called *free vectors*—vectors having the freedom to move from one location to another, so that $\overrightarrow{OP}$ and $\overrightarrow{QR}$ in Figure 2.2, for example, which have the same magnitude and the same direction, are officially counted not as distinct vectors but as the same free vector. Then we could indeed combine free vectors that were quite far apart by bringing them end-to-end, like $\overrightarrow{OP}$ and $\overrightarrow{PR}$ in Figure 2.2. But since we could also combine them according to the parallelogram law by moving them so that they have a common starting point, like $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in Figure 2.1, the parallelogram law is the basic one. Note that when we speak of the same direction we mean just that, and not opposite directions—north and south are not the same direction.

*Have you noticed that we have been careless in sometimes speaking of "the vector represented by $\overrightarrow{OP}$," at other times calling it simply "the vector $\overrightarrow{OP}$," and now calling it just "$\overrightarrow{OP}$"? This is deliberate—and standard practice among mathematicians. Using meticulous wording is sometimes too much of an effort once the crucial point has been made.

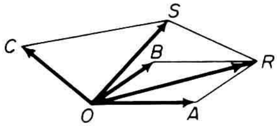
Exercise 2.6 Find the resultant of the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ in the diagram.



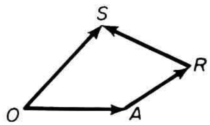
Solution We first form the resultant, $\overrightarrow{OR}$, of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ like this:



and then we form the resultant, $\overrightarrow{OS}$, of $\overrightarrow{OR}$ and $\overrightarrow{OC}$ like this:



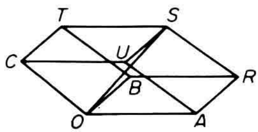
This figure looks complicated. We can simplify it by drawing only half of each parallelogram, and then even omitting the line OR, like this:



From this we see that the resultant $\overrightarrow{OS}$ can be found quickly by thinking of the vectors as free vectors and combining them by placing them end-to-end: $\overrightarrow{AR}$, which has the same magnitude and direction as $\overrightarrow{OB}$, starts where $\overrightarrow{OA}$ ends; and then $\overrightarrow{RS}$, which has the same magnitude and direction as $\overrightarrow{OC}$, starts where $\overrightarrow{AR}$ ends.

Exercise 2.7 Find, by both methods, the resultant of the vectors in Exercise 2.6, but by combining $\overrightarrow{OB}$ and $\overrightarrow{OC}$ first, and then combining their resultant with $\overrightarrow{OA}$. Prove geometrically that the resultant is the same as before.

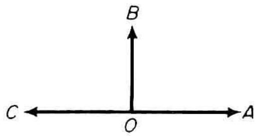
Exercise 2.8



The above diagram looks like a drawing of a box. Show that if we drew only the lines OA, AR, RS, and OS we would have essentially the last figure in Exercise 2.6; that if we drew only the lines OB, BT, TS, and OS we would have a corresponding figure for Exercise 2.7; and that if we drew only OA, AU, US, and OS we would have a figure corresponding to our having first combined $\overrightarrow{OA}$ with $\overrightarrow{OC}$ and then their resultant with $\overrightarrow{OB}$.

Exercise 2.9 In Exercises 2.6, 2.7, and 2.8, is it essential that the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ lie in a plane? Give a rule for finding the resultant of three noncoplanar vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ that is analogous to the parallelogram law, and that might well be called the parallelepiped law. Prove that their resultant is the same regardless of the order in which one combines them.

Exercise 2.10 Find the resultant of the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ below by combining them in three different orders, given that vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$ have equal magnitudes and opposite directions. Draw both the end-to-end diagrams and the full parallelogram diagrams for each case.



3. JOURNEYS ARE NOT VECTORS

It is all very well to start with a definition. But it is not very enlightening. Why should scientists and mathematicians be interested in objects that have magnitude and direction and combine according to the parallelogram law? Why did they even think of such objects? Indeed, do such objects exist at all —outside of the imaginations of mathematicians?

There are, of course, many objects that have both magnitude and direction. And there are, unfortunately, many books about vectors that give the reader the impression that such objects obviously and inevitably obey the parallelogram law. It is therefore worthwhile to explain carefully why most such objects do not obey this law, and then, by a process of abstraction, to find objects that do.

Suppose that I live at A and my friend lives 10 miles away at B. I start from A and walk steadily at 4 m.p.h. for $2\frac{1}{2}$ hours. Obviously, I walk 10 miles. But do I reach B?

You may say that this depends on the direction I take. But what reason is there to suppose that I keep to a fixed direction? The chances are overwhelming that I do not—unless I am preceded by a bulldozer or a heavy tank.

Most likely I walk in all sorts of directions; and almost certainly, I do not arrive at B. I may even end up at home.

Exercise 3.1 Where are all the possible places at which I can end, under the circumstances?

Now suppose that I start again from A and this time end up at B. I may take four or five hours, or I may go by bus or train and get there quickly. Never mind how I travel or how long I take. Never mind how many times I change my direction, or how tired I get, or how dirty my shoes get, or whether it rained. Ignore all such items, important though they be, and consider the abstraction that results when one concentrates solely on the fact that I start at A and end at B. Let us give this abstraction a name. What shall we call it? Not a "journey." That word reminds us too much of everyday life—of rain, and umbrellas, and vexations, and lovers meeting, and all other such items that we are ignoring here; besides, we want to preserve the word "journey" for just such an everyday concept. For our abstraction we need a neutral, colorless word. Let us call it a *shift*.

Here are routes of four journeys from A to B:

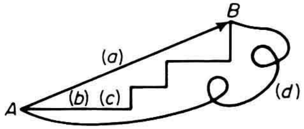


Figure 3.1

All four journeys are different—with the possible but highly improbable exception of (b) and (c).

Exercise 3.2 Why "highly improbable"?

But though the four journeys are not all the same, they yield the same shift. We can represent this shift by the arrow-headed line segment AB. It has both magnitude and direction. Indeed, it seems to have little else. Is it a vector? Let us see.

Consider three places A, B, and C as in Figure 3.2. If I walk in a straight

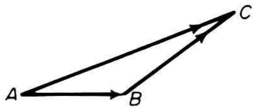


Figure 3.2

line from A to B and then in a straight line from B to C, I make a journey from A to C, but it is not the same as if I walked directly in a straight line from A to C: the scenery is different, and so is the amount of shoe leather consumed, most likely, and we can easily think of several other differences.

Exercise 3.3 Why "most likely"?

Thus, though we could say that the walks from A to B and from B to C combine to give a "resultant" journey from A to C, it is not a journey in a straight line from A to C: the walks do not combine in a way reminiscent of the way in which vectors combine; they combine more in the tautological sense that $2 + 1 = 2 + 1$ than $2 + 1 = 3$.

Journeys, then, are not vectors. But when we deal with shifts we ignore such things as the scenery and the amount of shoe leather consumed. A shift from A to B followed by a shift from B to C is indeed equivalent to a shift from A to C. And this reminds us so strongly of the vectorial situation in Figure 2.2 that we are tempted to conclude that shifts are vectors. But there is a crucial difference between the two situations. We cannot combine the above shifts in the reverse order (compare Exercise 2.5). There is no single equivalent to the shift from B to C followed by the shift from A to B. We can combine two shifts only when the second begins where the first ends. Indeed, in Figure 2.1, just as with journeys, we cannot combine a shift from O to P with one from O to Q in either order. Thus shifts are not vectors.

4. DISPLACEMENTS ARE VECTORS

Now that we have discovered why shifts are not vectors, we can easily see what further abstraction to make to obtain entities that are. From the already abstract idea of a shift, we remove the actual starting point and end point and retain only the relation between them: that B lies such and such a distance from A and in such and such a direction.* Shifts were things we invented in order to bring out certain distinctions. But this new abstraction is an accepted mathematical concept with a technical name: it is called a *displacement*. And it is a vector, as we shall now show.

In Figure 4.1, the arrow-headed line segments AB and LM are parallel and

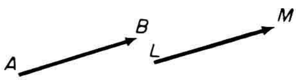


Figure 4.1

of equal length. Any journey from A to B is bound to be different from a journey from L to M. Also, the shift from A to B is different from that from L to M because the starting points are different, as are the end points. But the two shifts, and thus also the various journeys, yield the same displacement: if, for example, B is 5 miles north–northeast of A, so too is M 5 miles north–northeast of L, and the displacement is one of 5 miles in the direction north–northeast.

*We retain, too, the recollection that we are still linked, however tenuously, with journeying, for we want to retain the idea that a movement has occurred, even though we do not care at all *how* or under what circumstances it occurred.

Exercise 4.1 Starting from a point A, a man bicycles 10 miles due east to point B, stops for lunch, sells his bicycle, and then walks 10 miles due north to point C. Another man starts from B, walks 4 miles due north and 12 miles due east and then, feeling tired, and having brought along a surplus of travellers' checks, buys a car and drives 6 miles due north and 2 miles due west, ending at point D in the pouring rain. What displacement does each man undergo? [*Ans.* $10\sqrt{2}$ miles to the northeast.]

Now look at Figure 2.1. The shift from O to P followed by the shift from P to R is equivalent to the shift from O to R. The shift from P to R gives a displacement $\overrightarrow{PR}$ that is the same as the displacement $\overrightarrow{OQ}$. Therefore the displacement $\overrightarrow{OP}$ followed by the displacement $\overrightarrow{OQ}$ is equivalent to the displacement $\overrightarrow{OR}$.

Exercise 4.2 Prove, similarly, that the displacement $\overrightarrow{OQ}$ followed by the displacement $\overrightarrow{OP}$ is also equivalent to the displacement $\overrightarrow{OR}$.

Thus, displacements have magnitude and direction and combine according to the parallelogram law. According to our definition, they are therefore vectors. Since displacements such as $\overrightarrow{AB}$ and $\overrightarrow{LM}$ in Figure 4.1 are counted as identical, displacements are free vectors, and thus are somewhat special. In general, vectors such as $\overrightarrow{AB}$ and $\overrightarrow{LM}$ are not counted as identical.

5. WHY VECTORS ARE IMPORTANT

From the idea of a journey we have at last come, by a process of successive abstraction, to a specimen of a vector. The question now is whether we have come to anything worthwhile. At first sight it would seem that we have come to so pale a ghost of a journey that it could have little mathematical significance. But we must not underestimate the potency of the mathematical process of abstraction. Vectors happen to be extremely important in science and mathematics. A surprising variety of things happen to have both magnitude and direction and to combine according to the parallelogram law; and many of them are not at all reminiscent of journeys.

This should not surprise us. The process of abstraction is a powerful one. It is, indeed, a basic tool of the mathematician. Take whole numbers, for instance. Like vectors, they are abstractions. We could say that whole numbers are what is left of the idea of *apples* when we ignore not only the apple trees, the wind and the rain, the profits of cider makers, and other such items that would appear in an encyclopedia article, *but also ignore even the apples themselves*, and concentrate solely on how many there are. After we have extracted from the idea of apples the idea of whole numbers, we find that whole numbers apply to all sorts of situations that have nothing to do with apples. Much the same is true of vectors. They are more complicated than whole numbers—so