CALCULUS

of the elementary functions

Merrill E. Shanks • Robert Gambill

Calculus of the Elementary Functions

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The intent throughout is to present the theoretical structure as simply as possible without excessive concern about subtleties. Calculus, after all, is a certain kind of calculation; calculus is the solving of problems. And so the student must do many problems—about 1200 for the full year is minimal and about 1500 would seem about right.

The student should read the text with pencil and paper at hand. Not only may he need to fill in details of the text, but he may need to expand the Examples. The earlier Examples are usually fully expanded. Later Examples leave out steps that the student should be able to complete for himself.

We have tried to write a reasonably short book. Thick books discourage the reader and excess verbiage can obscure ideas. Moreover short exposition permits the teacher to expand on the text in his own style. Nevertheless, there is more than enough material here for a year's work even with well prepared students. The unstarred sections comprise a standard course. Selections can be made from the starred sections and problems to accommodate better students. These usually are either harder computation or more theoretical, and sometimes contain material given in advanced calculus.

Two other features are the exclusion of analytic geometry proper and the separation of differential and integral calculus. It seems to us that the monster texts that combine concepts of geometry, the derivative, and the integral into a unified whole present such a kaleidoscopic pattern that the student is often confused. In those books there may be one hundred or more pages devoted to pre-calculus topics including perhaps discussion of the real number system at a level for which the student sees no use. The pre-calculus background in this book is provided in Appendix A where the needed formulas and definitions are supplied for reference. The text proceeds at once with the calculus and soon the student is doing interesting problems. After all, there are but two basic techniques to be learned: differentiation and integration. Once these are mastered the student has powerful tools to attack problems which previously he could not touch. Too often students do not *feel* this gain in power. We think that in part this has been because they do not master one technique (differentiation) and learn to apply it before learning another.

Part I (Chapters 1 through 8) is devoted to the differential calculus of functions of one variable. The only mention of integration is a short section on antiderivatives in Chapter 1. This is for the benefit of students taking physics at the same time. However, if the instructor so desires, Chapter 9 on the definite integral can be taken up immediately following Chapter 4.

Part II (Chapters 9 through 13) is devoted to the integral calculus. It finishes with the theory of infinite series, which was treated in an informal intuitive way in Chapter 8.

Part III (Chapters 14 and 15) contains the standard topics of multivariate calculus. Although vectors are mentioned, no vector notation is used nor knowledge of vectors presupposed.

Rigorous proof has, on the whole, been relegated to the background and to later portions of the text. (Appendixes C and D are concerned with limits and continuity and some proofs of basic theorems.) We are much more concerned that a student have intuition about what is going on than that he remember proofs of analysis. But we are concerned that he understand the theorems and that he is able to verify the hypotheses in theorems and to apply them. Students arrive in college with such diverse backgrounds and attitudes that if one were to present calculus with complete rigor, one would have to provide the foundation for that rigor—and calculus proper would be delayed unduly. In the early chapters theorems about general functions are avoided where possible. In the early stages when one deals exclusively with the elementary functions general limit theorems are unnecessary, and to the student often seemingly irrelevant. A student needs only to "see" the particular limits that occur in treating elementary functions. The theorems that do occur are definitely utilized in problems—which is merely another way of saying that the text (except for historical remarks) is purposefully tied to expected student activity.

We have tried not to let the text get in the way of the students learning to calculate. The logical development is direct, and such that details and proofs of some theorems are easily supplied. We have tried to avoid the inclusion of anything that might have to be unlearned later. All that is here is usable.

West Lafayette, Indiana
January 1969

Merrill E. Shanks
Robert Gambill

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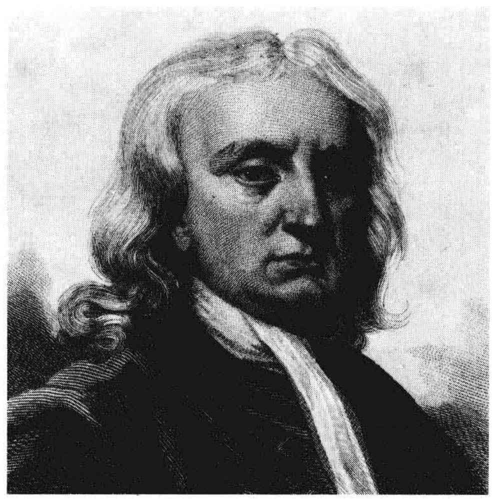
PART I

DIFFERENTIAL CALCULUS



Sir Isaac Newton
(1642–1727)
Born December 25, 1642 in Woolthorpe
Entered Trinity College
Cambridge 1661
Undergraduate degree January, 1664
At home 1665–1667
Fellow at Trinity 1667–1669
Professor of mathematics, Lucasian chair,
at Trinity 1669–1696
Publishing date: Philosophiae Naturalis
Principia Mathematica (Mathematical Principles
of Natural Philosophy) 1687
Warden of the London Mint, 1696
Master of the Mint 1699–1727

Newton is generally regarded as one of the great intellects of all time. His influence on the development of mathematics and physics was decisive. He entered Cambridge knowing almost no mathematics but advanced rapidly under the brilliant Isaac Barrow, his teacher.

The two years spent at home during a recurrence of the bubonic plague were fantastically productive. During this time: (1) He invented the calculus, which he called the method of *fluxions,* and had it in fairly complete form—for his own use. (2) He conceived the principle of universal gravitation and sketched its main outlines. (3) He discovered the decomposition of white light into a spectrum of colors and devised optical equipment.

On his return to Cambridge the pupil soon surpassed the teacher, and Barrow resigned his Lucasian chair of mathematics in favor of Newton. There a long and productive period ensued in which he published little material except at the urging of his friends. News of his activity was known in England mainly through letters and the words of friends, while the continent remained unaware of the method of fluxions.

CHAPTER 1

THE DERIVATIVE

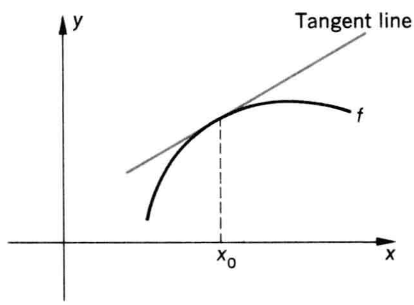
1 The Problem of Tangents

In the seventeenth century mathematicians were concerned (among other things) with two major problems, "The Problem of Tangents" and "The Problem of Area," whose nature and significance will become apparent as the reader progresses with the text. The area problem will be introduced in Chapter 9 of Part II. Part I is devoted to the solution of the Problem of Tangents and its application to a variety of problems.

Actually there are two aspects to "The Problem of Tangents."

(1) Given a function f what should we *mean* by a tangent line to the graph of f?

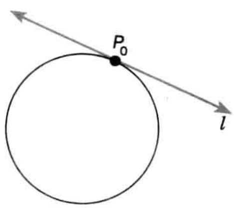
(2) Is there an easy way, a trick, for calculating what the tangent is?



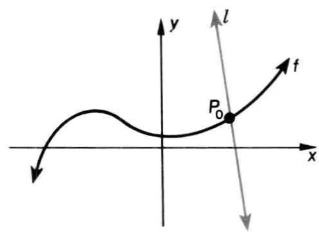
In this chapter we are concerned only with question (1). Chapter 2 deals with and solves question (2). The reader may be surprised, as he proceeds, with the apparent simplicity of the solutions to both problems. But in the seventeenth century the problems were obscured by several deficiencies. In the first place suitable notation (that marvelous shorthand that presents ideas concisely) was not well developed. Secondly, the real number system was not adequately formulated. (Not until the nineteenth century did Dedekind give a modern construction of the real numbers.) But the main obstacle to a solution to problem (1) was the fact that the central concept required the notion of *limit*—and limits were not properly understood. This, too, was to come later.

It is hard for the beginner to understand why calculus was not invented earlier. Indeed, Isaac Barrow (the teacher of Newton) and the great Pierre Fermat were aware of all the pieces of both problems, namely Tangents and Area. But the fact remains that it was left to the genius of Newton and Leibniz to show the way to handle both problems. [The historically interested reader is urged to consult the references at the end of the chapter.]

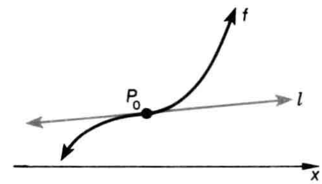
There *is* a question as to what the definition should be. Observe:



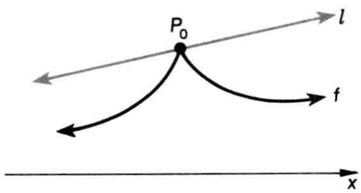
In plane geometry the tangent line meets the circle in exactly one point



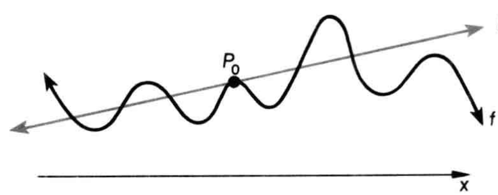
The intersection is one point yet l is hardly tangent at P_0. Line l crosses the graph



Here line l crosses the graph and nevertheless is the tangent at P_0

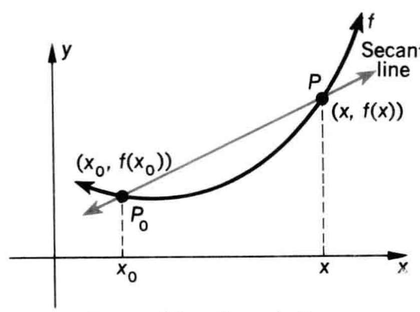


Is line l tangent?



Here l is tangent at P_0 yet meets the graph often

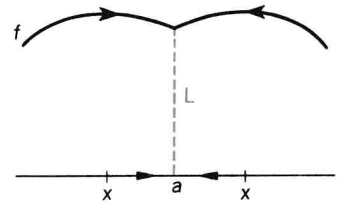
The basic definition gives the tangent line as a *limit* of secant lines. First we need a word of explanation of the idea of limit.



A secant line through P_0

We say that a function f has a limit* L as x approaches a if as x gets closer to a (that is, $|x - a|$ gets smaller, *but does not become zero*), then $f(x)$ gets closer to the number L (that is, $|f(x) - L|$ gets arbitrarily small). This is expressed symbolically by

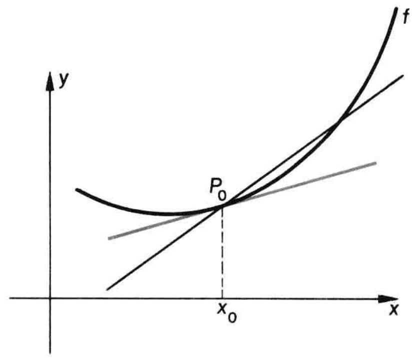
$$\lim_{x \to a} f(x) = L.$$



$f(x)$ approaches L as x approaches a, or $f(x) \to L$ as $x \to a$

DEFINITION The tangent line to the graph of f at $P_0 = (x_0, f(x_0))$ is the line through P_0 whose slope is the limit of the slopes of the secant lines through $(x_0, f(x_0))$ and $(x, f(x))$ as x approaches x_0.

Remark. This definition does not permit vertical tangent lines, because in that case the limit of the slopes of the secant lines does not exist. Example 3 below illustrates this possibility.



The tangent line is determined as soon as we know its slope, m, for then it is the line with slope m passing through $(x_0, f(x_0))$.

The slope of a secant line is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

So as x approaches x_0 this number should approach the slope of the tangent line. We formulate this in terms of "increments," a notation terminology that was devised by Leibniz.

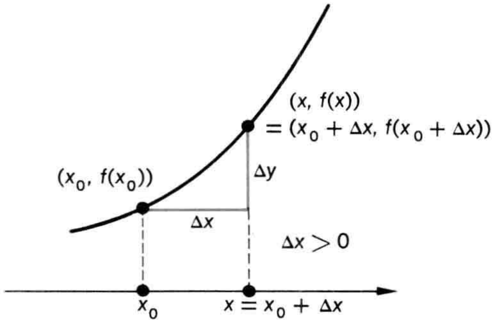
$$\begin{aligned} \Delta x &= x - x_0 = \text{increment}\dagger \text{ of } x; \\ \Delta y &= y - y_0 \\ &= f(x) - f(x_0) \\ &= f(x_0 + \Delta x) - f(x_0) \\ &= \text{increment of } y. \end{aligned}$$

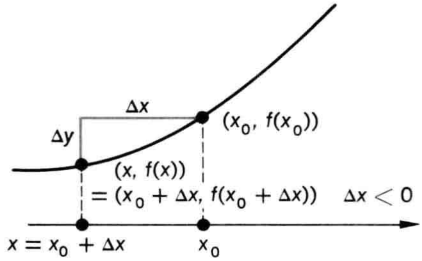
---

* We shall not try to be absolutely precise about limits in the text proper. Our concern is with an intuitive grasp of the concepts. Moreover, in the special cases that concern us, the required limit will be rather obvious because of the simplicity of the algebra involved.

A precise definition of limit and some theorems about limits can be found in Appendix C, Section 1.

† The symbol "Δx," read "delta-x," is a single quantity. Thus Δx^2 will mean $(\text{delta-}x)^2 = (\Delta x)^2$. Δx can be either positive or negative. If Δx is negative, the point x lies "to the left" of x_0.





Then the slope of the secant line is

$$m_{\text{secant}} = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

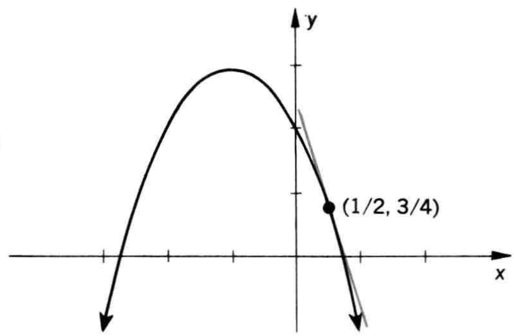
In particular cases we can see that this slope approaches a limit as Δx approaches 0. This limit is then the slope of the tangent line.

Example 1 If $y = f(x) = 2 - 2x - x^2$, then the slope of the tangent line at $(\frac{1}{2}, \frac{3}{4})$ is the limit of slopes of secant lines.

Here $x_0 = \frac{1}{2}$, $y_0 = \frac{3}{4}$, and

$$\begin{aligned} \Delta y &= f(x_0 + \Delta x) - f(x_0) \\ &= f(\tfrac{1}{2} + \Delta x) - \tfrac{3}{4} \\ &= 2 - 2(\tfrac{1}{2} + \Delta x) - (\tfrac{1}{2} + \Delta x)^2 - \tfrac{3}{4} \\ &= -3\Delta x - \Delta x^2. \end{aligned}$$

$$\frac{\Delta y}{\Delta x} = -3 - \Delta x.$$



And *now* it is easy to see what happens to $\Delta y/\Delta x$ as $\Delta x \to 0$. Observe that both Δy and Δx approach 0, so we could not have evaluated the limit simply by setting $\Delta x = 0$ and $\Delta y = 0$ for then we would have obtained the meaningless symbol 0/0. Instead, we have so simplified the form of $\Delta y/\Delta x$ that we can *see* at once its limit as $\Delta x \to 0$. Clearly the limit is -3, and so

$$m = \text{slope of the tangent line at } (\tfrac{1}{2}, \tfrac{3}{4}) = -3,$$

and an equation of the tangent line is:

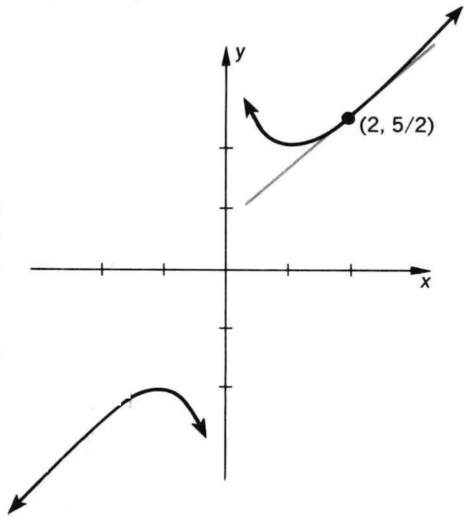
$$y - \tfrac{3}{4} = -3(x - \tfrac{1}{2}) \qquad \text{or} \qquad y + 3x - \tfrac{9}{4} = 0.$$

Example 2 In the Example 1 we found the slope of the tangent at a specific point. Usually we find the slope at an arbitrary point $(x, f(x))$. In this way we obtain a formula for the slope at any point. For example, if

$$y = f(x) = x + \frac{1}{x},$$

then $$\Delta y = f(x + \Delta x) - f(x) = \left(x + \Delta x + \frac{1}{x + \Delta x}\right) - \left(x + \frac{1}{x}\right)$$

$$= \Delta x + \frac{1}{x + \Delta x} - \frac{1}{x} = \Delta x + \frac{-\Delta x}{x(x + \Delta x)}.$$



Hence,

$$\frac{\Delta y}{\Delta x} = 1 - \frac{1}{x(x + \Delta x)}.$$

Again, as in Example 1, we have so simplified $\Delta y/\Delta x$ that its limit as Δx approaches 0 is clear. The slope of the tangent at $(x, x + 1/x) = m = 1 - 1/x^2$. For example, the slope at $(2, \frac{5}{2})$ is $m = \frac{3}{4}$, and an equation of the tangent line at this point is:

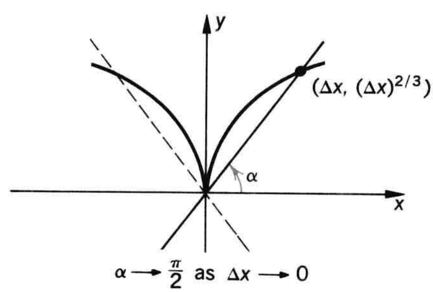
$$y - \tfrac{5}{2} = \tfrac{3}{4}(x - 2)$$

or

$$y - \tfrac{3}{4}x - 1 = 0.$$

Example 3 A secant line to $y = f(x) = x^{2/3}$ at $(0, 0)$ has slope

$$m = \frac{\Delta y}{\Delta x} = \frac{(\Delta x)^{2/3}}{\Delta x - 0} = (\Delta x)^{-1/3}.$$



Now, as $\Delta x \to 0$ the slope does not approach a limiting value. Nevertheless, the inclination α of the secant line approaches $\pi/2$.

In such examples we agree that the tangent line at $(x_0, f(x_0))$ is the vertical line $x = x_0$. In this example $x_0 = 0$.