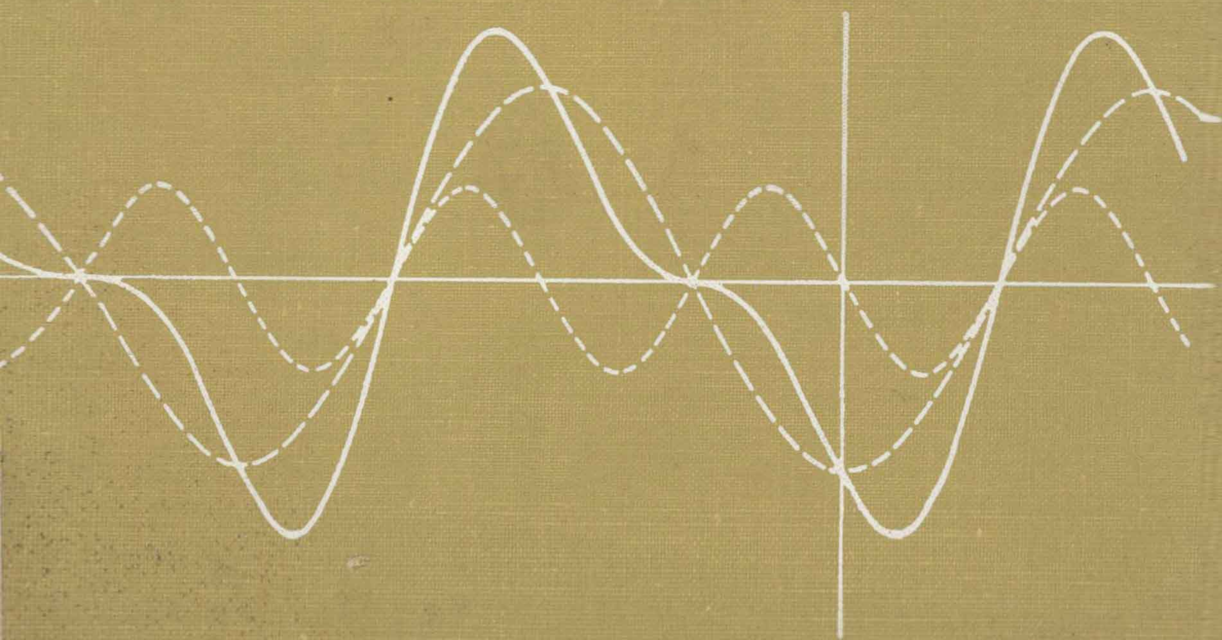
FRED RICHMAN
CAROL WALKER
ELBERT WALKER

COLLEGE TRIGONOMETRY



FRED RICHMAN
CAROL WALKER
ELBERT WALKER
New Mexico State University

COLLEGE TRIGONOMETRY

SCOTT, FORESMAN AND COMPANY

Library of Congress Catalog Number 69-15228

Regional Offices of Scott, Foresman and Company are located in Atlanta, Dallas, Glenview, Palo Alto, Oakland, N.J. and London, England.

College Trigonometry

Preface

This book was written with the idea of providing an uncluttered and mathematically sound introduction to the basic ideas of trigonometry, emphasizing the important concepts, without attempting to initiate the student into the mysteries of higher algebra and set theory. To this end, we combine the classical, intuitive approach with the important facets of the modern point of view.

Trigonometry is viewed as the study of the trigonometric functions. The general notion of a function is developed with stress placed upon understanding rather than set theoretical niceties. Real numbers are placed in their natural setting—as the results of measurements. These two topics are treated in some detail at a concrete level, to lay the foundation not only for the rest of the text but also for the calculus. Complex numbers are introduced as a natural extension of the real number system, without the complication of an artificial construction by way of ordered pairs.

The trigonometric functions are introduced via the trigonometric point. This is the "modern way," and seems to have some mathematical if not pedagogical advantages over the classical one. In line with this approach, the addition formulas are motivated and developed by examining rotations of the plane. Angles are not ignored, however, and right triangle trigonometry plays its just role.

In the spirit of the contemporary functional approach, logarithms are developed from consideration of the inverse of the exponential function, and the inverse trigonometric functions from consideration of the inverses of portions of the trigonometric functions.

For the benefit of the student, many examples are provided, and answers are given for odd-numbered problems.

Las Cruces, New Mexico

Fred Richman
Carol Walker
Elbert Walker

Contents

1 *The Real Numbers*

1–1 Introduction

Historically trigonometry is the study of triangles, particularly right triangles, and the relations between the lengths of their sides and the sizes of their angles. It was developed mainly for use in surveying and astronomy, to help measure the earth and the sky. In the study of right triangles certain numbers came to be associated with angles. These numbers did not directly measure the size of the angles but rather the relationships between the sides of any right triangle having those angles. Thus the notion of a trigonometric function was born.

The child has outgrown the parent in this case, since the trigonometric functions have come to have far-reaching applications—in engineering, physics, and higher mathematics—that completely overshadow their use in the study of triangles. Even the very notion of trigonometric functions no longer depends on their right triangle genealogy, since they may be thought of purely in terms of numbers and not in terms of angles. Thus trigonometry has come to mean the study of the trigonometric functions, and this is the point of view that we shall take: the fundamental objects of study in this book are the trigonometric functions, not triangles. However, the important connections between these functions and triangles are not to be ignored, and they will be brought out in Chapters 4 and 8.

Since our concern is with trigonometric functions, and since these are functions of real numbers, we need to know what real numbers are and what functions are. This chapter presents a quick review of some of the basic properties of real numbers.

1–2 Measuring

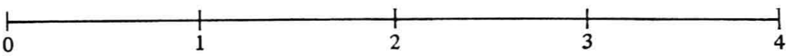
The numbers we use to measure things are called *real* numbers. In order to measure lengths we first decide upon a *unit length* which we represent by the number 1. All lengths are measured in terms of the unit length. This unit length might be an inch, a foot, a mile, a light year, or whatever. Suppose, for example, we have chosen an inch-long segment as our unit length. Here is our unit length.

(1)

Now suppose we are faced with another length:

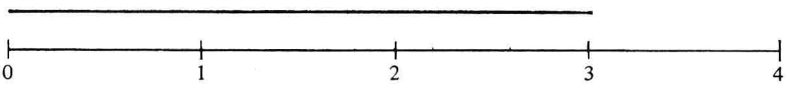
(2)

How do we measure it? We construct a "ruler" by marking off a line at intervals of one inch.



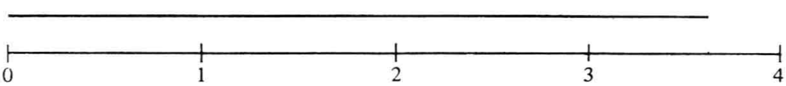
(3)

To measure a length we line up one end with the point marked 0 on our ruler and see where the other end lands. Thus to measure the segment in (2) we simply place our ruler next to it and verify immediately that it is three inches long. The numbers 0, 1, 2, 3, ... labeling these marks are called *integers*.



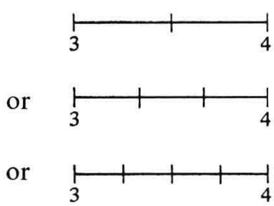
(4)

What happens if the length we are trying to measure falls somewhere between two of the marks on our ruler? For example,



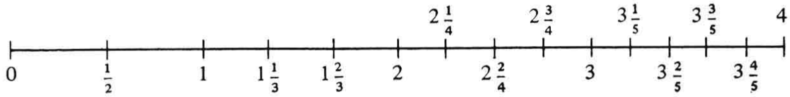
(5)

The length is somewhere between three and four inches. If we wish to be more precise, we shall have to put more marks on our ruler. One way is to divide the interval from 3 to 4 into equal sections like



(6)

In the first case we label the new point "$3\frac{1}{2}$," since we divided the segment into two equal sections. The points in the second case would be labeled $3\frac{1}{3}$ and $3\frac{2}{3}$, since the segment was divided into three equal sections. Similarly, the points in the last case would be labeled $3\frac{1}{4}$, $3\frac{2}{4}$, and $3\frac{3}{4}$. Notice that the point labeled $3\frac{2}{4}$ is the same one labeled $3\frac{1}{2}$. These two labels, $3\frac{2}{4}$ and $3\frac{1}{2}$, are different ways of referring to the same length. We may continue in this way to mark our ruler at all of the points which can be labeled by fractions. A few are pictured below.

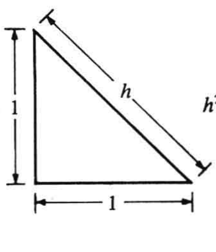


(7)

The numbers used to label these marks—the integers and the fractions—are called *rational* numbers. The distinguishing feature of a rational number is that it can be written as a quotient a/b with a and b integers. For example,

$$6\frac{1}{8} = \frac{49}{8}, \quad 3\frac{1}{2} = \frac{7}{2}, \quad \text{and} \quad 4 = \frac{4}{1}.$$

With the rational numbers we can measure many more lengths than we could with integers alone. Can we measure *all* lengths? This question provoked a stormy controversy among the ancient mathematicians. Pythagoras was reputed to have shown that if a right triangle had sides which were one inch long, then the square of the length of the hypotenuse was equal to 2, the sum of the squares of the lengths of the two sides.

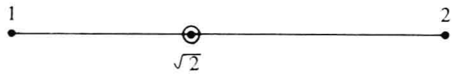


$$h^2 = 1^2 + 1^2 = 2 \qquad (8)$$

If we could measure the hypotenuse of this triangle with our ruler which is marked only with rational numbers, then we could write h as a/b, where a and b were integers. This would give us integers a and b, such that $(a/b)^2 = 2$. However, the ancients were able to show that this is impossible. Hence this real number, which we write $\sqrt{2}$ and call "the square root of 2," is not yet marked on our ruler. If we are to measure everything, we shall have to put it on the ruler, along with a host of other numbers which are missing. These missing numbers are called *irrational* numbers. The numbers $\sqrt{2}$ and π are examples of such numbers. Their distinguishing feature is that they *cannot* be written as a quotient a/b with a and b integers.

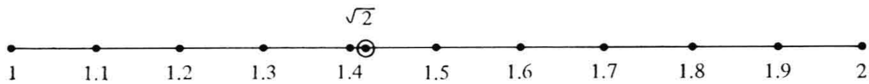
1–3 *Decimal Representation*

Decimal representation, or expansion, is a unifying technique for labeling real numbers. All of the real numbers, both rational and irrational, have a decimal representation. To see how it works, let us consider the number $\sqrt{2}$. Since the square of $\sqrt{2}$ lies between the square of 1 and the square of 2 ($1^2 = 1$, $(\sqrt{2})^2 = 2$ and $2^2 = 4$), we know that $\sqrt{2}$ must lie somewhere between 1 and 2.



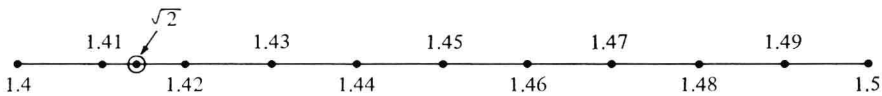
(9)

To get a better idea of where $\sqrt{2}$ is, we divide the interval from 1 to 2 into ten equal segments (hence the terminology "decimal," from the Latin word for ten).



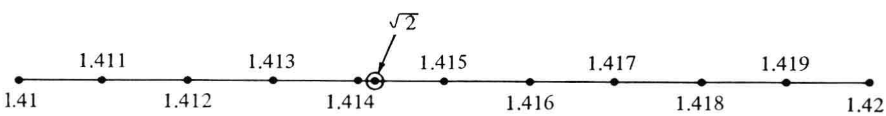
(10)

We label the division points 1.1, 1.2, 1.3, etc., and observe that, in the fraction notation, these are the points $1\frac{1}{10}$, $1\frac{2}{10}$, $1\frac{3}{10}$, and so on. We see that $\sqrt{2}$ lies between 1.4 and 1.5. This is easily verified by checking that $(1.4)^2 = 1.96$ which is less than 2, while $(1.5)^2 = 2.25$ which is greater than 2. Now, breaking the interval from 1.4 to 1.5 into ten equal parts,



(11)

we find that $\sqrt{2}$ lies between 1.41 and 1.42, the points labeled $1\frac{41}{100}$ and $1\frac{42}{100}$ by the fractions. Once more we divide into ten equal pieces



(12)

and find that $\sqrt{2}$ lies between 1.414 and 1.415. Continuing in this fashion we may construct an infinite decimal expansion 1.414213.... This expansion means that $\sqrt{2}$ is at least as big as any of the numbers

$$1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, \ldots$$

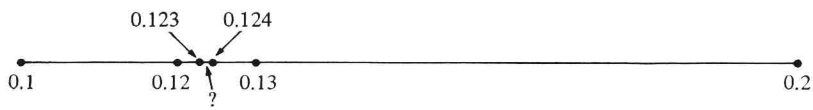
but no bigger than any of the numbers

$$2, 1.5, 1.42, 1.415, 1.4143, 1.41422, 1.414214, \ldots.$$

Similarly, to say that the decimal expansion of π is 3.14159265358979... simply means that π lies between 3 and 4, between 3.1 and 3.2, between 3.14 and 3.15, between 3.141 and 3.142, between 3.1415 and 3.1416, and so on.

If you know the first few digits in the decimal representation of a number, you know approximately what that number is. For example, if the decimal representation of a number x starts out 3.1652, then x can be no less than 3.1652 and no greater than 3.1653. Indeed, the way the fourth digit 2 was determined was by dividing the interval from 3.165 to 3.166 into ten equal pieces and observing that x lay between the points labeled 3.1652 and 3.1653. Approximations will be discussed in more detail in Section 1–5.

To every point on our ruler we have made correspond a possibly infinite decimal representation. On the other hand, suppose we are given an infinite decimal, for example 0.123456789101112131415161718192021222324252.... (Mathematicians are fond of using three dots to mean "etc." when the reader should know what follows, e.g., 1, 2, 3, 4, 5, ... or 2, 4, 8, 16, 32, 64, 128, ..., or in the example above which is obtained from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, ... by removing commas and spaces.) Is there some point on our ruler which is labeled by this infinite decimal? That is, is there a point which is at once between 0.1 and 0.2, between 0.12 and 0.13, between 0.123 and 0.124, between 0.1234 and 0.1235 and so on? There is no pressing geometric reason for such a point to exist, as there was for 1.414213.... Yet somehow we feel (or perhaps you don't) that there is some point which separates the points 0.1, 0.12, 0.123, 0.1234, 0.12345, ... from the points 0.2, 0.13, 0.124, 0.1235, 0.12346, ...



(13)

This point would be labeled by the infinite decimal 0.1234567891011.... We shall adopt the view that indeed such a point exists and that there is a point on our ruler corresponding to any decimal expansion. To do so does not offend common sense at least and is a matter of great convenience.

One last thing needs to be cleared up before we identify the real numbers with the decimal representations of lengths. What is the point represented by 1.9999999... ? This point is between 1 and 2, between 1.9 and 2, between 1.99 and 2, between 1.999 and 2, and so forth. The *only* point it could possibly be is the point labeled 2. Thus we have two different decimal representations of the point 2. Similarly, 1.569999999... *must* label the same point as 1.57. With this in mind we can identify real numbers (or, looking ahead, non-negative real numbers) with possibly infinite decimals and say that two real numbers are equal, if the decimals are the same or if they are related to each other as are, for example, 1.4329999999... and 1.433.

Problems 1–3

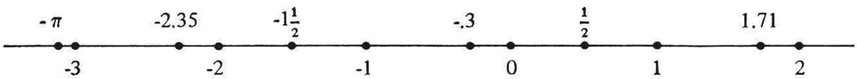
1. If the decimal expansion of a real number begins with 2.3156, what can you say about the number?
2. If the decimal expansion of a real number begins with 3.1427, what can you say about the number?
3. The first five digits in a decimal representation of x are 0.76548; the first four digits in a decimal representation of y are 0.7655. What can you say about y as compared to x? Why?
4. The first five digits in a decimal representation of x are 0.35916; the first four digits in a decimal representation of y are 0.3591. What can you say about y as compared to x? Why?
5. The first four digits in a decimal representation of x are 0.2719; the first four digits in a decimal representation of y are 0.2720. What can you say about y as compared to x? Why?
6. The first three digits in a decimal expansion of x are 0.357; the first three digits in a decimal expansion of y are 0.356. What can you say about y as compared to x? Why?
7. Show that the first four digits in a decimal representation of $\sqrt{7}$ are 2.645.
8. Show that the first seven digits in a decimal representation of $\sqrt{2}$ are 1.414213. Why do some tables give $\sqrt{2}$ as 1.414214?

1–4 Negative Numbers, Absolute Value, and Order

Walk 5 miles north and 6 miles south; where are you? You earn \$110 and spend \$125; how much money have you accumulated? The temperature was 5° and dropped 7°; what is the temperature now? Questions like these led to the development of the notions of "directed distances" and "signed numbers." The real numbers that we have looked at so far did not appear quite adequate for measuring along a line; not only do you want to know *how far away* a point is but *in what direction*. We talk about 1 mile *south*, \$15 *in the red*, or 2° *below zero*.

An efficient procedure for indicating direction along with distance is to choose one direction to be *positive* (e.g., north, gain as opposed to loss, above zero) and measure distances or amounts in that direction as we did before. The other direction we call the *negative* direction. We measure distances that way with numbers that are somehow distinguished: by writing them with red ink, for example, or, as is most commonly done, by prefixing them with a dash (minus sign) as -3, -7.2, $-1/2$, $-\pi$. In this context numbers represent two things, a distance and a direction, the direction being indicated by the presence or absence of a minus sign.

We can extend our ruler to enable us to measure in both directions. This extended ruler is often referred to as the *number line*. It looks like this:



(14)

The points on the number line correspond to the real numbers, positive, negative, and zero. The negatives of integers are also called integers; the negatives of rational numbers are rational numbers, and the negatives of irrational numbers are irrational.

The *absolute value* of a real number is the distance it represents, regardless of direction. If a is a real number, we denote its absolute value by $|a|$. Thus $|2| = 2$, $|-3| = 3$, $|-2/3| = 2/3$, and so on. If we think of the real numbers as being points on a line, then $|a|$ is simply the distance from a to 0. More generally, a simple geometric interpretation is available for $|a - b|$: it is the distance between a and b. A few examples illustrate this: $|3 - 5| = 2 =$ distance between 3 and 5; $|3 + 5| = |3 - (-5)| = 8 =$ distance between 3 and -5; $|-3 - 5| = 8 =$ distance between -3 and 5.

An important property of the real numbers is that they are *ordered*; i.e., we know what it means for a number to be bigger than another. If a is bigger than b, we write $a > b$ or $b < a$ (the smaller part of the symbol " $>$ " points to the smaller number). So $5 > 3$, $2 < \pi$, and so on. What about -1 and -1000? If I am \$1 in debt I have more money than if I am \$1000 in debt. Hence we write $-1 > -1000$. The rule is: $a > b$ if a lies to the right of b on the number line. Hence a number a is positive if $a > 0$ and negative if $a < 0$.

The notion of "positiveness" is the key to the notion of order. We have $6 > 2$ because $6 = 2 + 4$ and 4 is positive. If you add a positive number to 2, you get something bigger than 2; conversely, if $a > 2$, then $a - 2$ is positive and $a = 2 + (a - 2)$. In general we can say that

$$a > b \text{ if and only if } a - b \text{ is positive.} \tag{15}$$

For example, $5 > 1$ because $5 - 1 = 4$ is positive; $-3 > -4$ because $-3 - (-4) = -3 + 4 = 1$ is positive.

If we write $a < b$, we mean that a is less than b; in particular we deny that a and b are equal. Thus it is not true that $5 < 5$. If we wish to include the possibility that a and b are equal, we write $a \leq b$ which is read, "a is less than or equal to b." The symbol $\leq$ is a combination of $<$ and $=$, which serves to remind us what it means. Similarly, "$a \geq b$" means "$a > b$ or $a = b$."

Problems 1–4

1. Arrange the following numbers in ascending order.
   $1, -3, \pi, \sqrt{2}, 3/5, -1.4, 2.7, -5/3$.
2. Arrange the following numbers in ascending order.
   $1.6, 0, -2, \pi/2, -9/5, 5/3, -\sqrt{2}$.
3. List all numbers whose absolute value is 17.
4. List all numbers whose absolute value is
   a) $1/2$, b) 0, c) -9.
5. Is the statement, "If $a > b$, then $a + x > b$" true or false? Why?
6. Discuss the following statements.
   a) If $a > |b|$, then $a > b$.
   b) If $|a| > |b|$, then $a > b$.
   c) If $a > b$, then $a + x > b + x$.
7. Show that $-3 > -5$, using the criterion that $a > b$ exactly when $a - b$ is positive.
8. Verify or deny the following statements using the criterion that $a > b$ exactly when $a - b$ is positive.
   a) $3 > -2$, b) $2 > -3$, c) $2 > |-3|$,
   d) $-5 > -7$, e) $-8 > 4$, f) $3 > -5$.

1–5 Rounding and Significant Digits

In practice we do not deal with numbers like 3.14159265358979.... Suppose we have a wheel whose diameter is 5 feet and wish to know its circumference. Now we know that the circumference is 5π. However, in all probability we really don't know whether the diameter of the wheel is 5 feet or, say, 5.0000001 feet (and we might not care even if we did). In this situation it would be pointless to use all digits in the expansion of π, even if this were possible, to compute the circumference. Depending on how much precision we demanded, and could use, we would approximate π by 3.14 or 3.1416 or 3.14159, and so on. This process is known as *rounding* or *rounding off.*

To round off π *to two decimal places* is to find a number with *two* digits after the decimal point, which is as close as possible to π. Since we know that π lies between 3.14 and 3.15 (why?), we need only determine which of these numbers is closer to π. But we also know that π lies between 3.141 and 3.142, both of which are closer to 3.14 than to 3.15. Hence π is closer to 3.14 than to 3.15 and thus 3.14 is as close as we can come to π using numbers with two digits after the decimal point. We say that π *is* 3.14 *to two decimal places.*

Similarly, we say that π is 3.1416 *to four decimal places*, since 3.1416 is as close as we can come to π using numbers with *four* digits after the decimal point. Here we know that π is between 3.1415 and 3.1416. To decide which of these numbers is closer to π we look at one more place in the decimal expansion. This tells us that π lies between 3.14159 and 3.14160, and thus 3.1416 is a better approximation than 3.1415.

Notice what we do in these two examples. In the first, to round π to *two* decimal places we look at the first *three* places in the expansion of π,

$$3.141.$$

Since the third digit after the decimal point is 1, we see that π is 3.14 to two decimal places. Similarly, all of the numbers

$$3.140,\ 3.141,\ 3.142,\ 3.143,\ 3.144$$

are 3.14 to two decimal places and so is any number whose decimal expansion starts out in any of these ways. (Why?)

On the other hand, the numbers

$$3.146,\ 3.147,\ 3.148,\ 3.149$$

are clearly closer to 3.15 than to 3.14, as is any number whose decimal expansion begins in one of these fashions. The key as to whether to round *down* to 3.14 or to round *up* to 3.15 lies in the *third* digit. If this is less than 5 we round *down*; if it is greater than 5 we round *up*. When we rounded π to *two* decimal places, we rounded *down* to 3.14 because the *third* digit is less than 5. When we rounded to four decimal places, we rounded up to 3.1416 because the *fifth* digit is greater than 5.

What if the crucial digit is 5? This occurs, for example, if we wish to round π to *three* places. Here we must choose between 3.141 and 3.142. If we go to four places, we have 3.1415 which is equally close to 3.141 and 3.142. When this happens we must consider the rest of the expansion. If *all* the remaining digits were 0, for example 3.1415000000000..., then the number is precisely half way between 3.141 and 3.142. In this situation there is no reason to choose the one approximation over the other. One rule of thumb, which has the advantage of overestimating as often as underestimating, is to round to the number which ends in an even digit, in this case to 3.142. However, if *any* of the remaining digits in the expansion are different from 0, as is the case for π, then you should round *up*. For example, 3.141500000100 . . . , since it is greater than 3.1415, is closer to 3.142 than to 3.141 (although not very much so).

We summarize these ideas with a fresh example. Suppose we wish to round to *two* decimal places. If the first *three* places in the expansion are

$$7.160, 7.161, 7.162, 7.163, \text{ or } 7.164,$$

we round to 7.16. If the first three places are

$$7.166, 7.167, 7.168, \text{ or } 7.169,$$

we round to 7.17. If the first three places are 7.165, we round to 7.17 if there are *any* nonzero digits in the remainder of the expansion. If, on the other hand, the number is *precisely* 7.165 then we (may) apply our rule of thumb and round to 7.16 since this ends in an even digit.

A similar procedure applies to rounding off a number in any place. Consider the number 567.8962. Rounded off to three decimal places, or in the third decimal place, or in the thousandths place, 567.8962 is 567.896. Rounded off to two decimal places, or in the hundredths place, 567.8962 is 567.90; rounded to one decimal place, or in the tenths place, it is 567.9. We may also round 567.8962 in the *units* place, that is, to the nearest integer. Here we would get 568 since that is the closest integer to 567.8962. Similarly, we may round 567.8962 to the tens place, that is, to the nearest integer which is a multiple of ten; if we do this we get 570. In the same spirit we say that 567.8962 is 600 to the nearest hundred, 1000 to the nearest thousand, and 0 to the nearest ten thousand; these statements mean that 567.8962 is nearer to 600 than to 500 or 700, nearer to 1000 than to 0 or 2000, and nearer to 0 than to 10,000.

Computation with approximate numbers, that is, numbers which are approximations, often yields results which appear more precise than they actually are. Let us return to our 5-foot diameter wheel. Suppose we measured the diameter as carefully as we could and found it was, in fact, 5.03 feet to within a hundredth of a foot; that is, we know that 5.03 is closer to the true diameter of the wheel than is either 5.02 or 5.04—and that is the full extent of our knowledge. In computing the circumference we might use 3.1416 as an approximation to π. Then we would estimate the circumference to be $(3.1416) \cdot (5.03) = 15.742248$. Notice that we could only measure the diameter up to one hundredth of a foot while our estimate of the circumference seems to be accurate up to a millionth of a foot! The problem is that the last few digits in our solution have no significance. Indeed, for all we know, the diameter might be anywhere from 5.025 to 5.035. If it were the former, we would compute the circumference to be $(3.1416) \cdot (5.025) = 15.72654$; whereas if it