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Preface

For the last twenty years a course in mathematical methods in engineering has been taught by the authors and by others at Princeton University. Although intended primarily for first year graduate students in Civil, Chemical and Mechanical Engineering, this course has been taken in recent years by engineering seniors and graduate students from science departments. This book has evolved from the notes prepared for the course by the first author and revised and modified by the second author during his sabbatical leave at Princeton. The chapter on numerical methods was prepared by the third author.

This is a book about computational mathematics for engineers. The motivation for the choice of subjects is their usefulness in engineering. The level of mathematical rigor is kept to the minimum necessary for the correct presentation of the material. This book is unique in the sense that it brings together under one cover analytical, approximate and numerical methods for the solution of engineering problems.

The presentation of material assumes that the reader has been through introductory courses in linear algebra, vector analysis, elementary differential equations and complex variables. However, these should not be considered as strict prerequisites. The reader with more background will often benefit from a review of the material. The reader lacking familiarity with one of the above subjects, such as complex variables, will be able to acquire the necessary background by extra work and outside reading.

The first four chapters cover a variety of background material including series, integration in the complex domain, linear algebra and an introduction to tensor calculus. The fifth chapter introduces integral equations and concepts from calculus of variations. Chapters six and seven cover ordinary differential equations with an introduction to approximate techniques such as perturbation methods. Chapter eight covers partial differential equations. Chapter nine introduces the concept of numerical methods for the solution of ordinary and partial differential equations such as finite difference, finite element and boundary element methods.

A number of problems are included at the end of each chapter. These are an integral part of the course for which this book is designed, as the emphasis is on understanding by means of examples. The material in this textbook can be covered in a year-long course with the first five chapters comprising the first term and the last four chapters the second term. Alternatively, the last four chapters only may be used as a course on differential equations because the material is self-consistent.

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Chapter 1

Series

1.1. Introduction

In this chapter we will introduce the concept of a series and its convergence. In addition, several tests will be introduced to determine the convergence of series. We will next introduce a different kind of series known as an asymptotic series. This series, though very useful in many applications, is not a convergent series. This series will be used to obtain approximate evaluation of integrals.

1.2. Convergent Series

1.2.1. Introduction and Definitions

A series is a sum of terms which may be finite, $a_1 + a_2 + \cdots + a_n$ or infinite $a_1 + a_2 + \cdots + a_n + \cdots$. The sequence

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= s_1 + a_2 = a_1 + a_2 \\ s_3 &= s_2 + a_3 = a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= s_{n-1} + a_n = a_1 + a_2 + \cdots + a_n \end{aligned} \tag{1.2-1}$$

is called partial sums.

If this sequence of partial sums has a limit as $n \to \infty$, that is

$$\lim_{n \to \infty} s_n = S \tag{1.2-2}$$

exists, then the series is said to be convergent. If S does not exist, then the series is said to be divergent and no numerical value may be associated with it.

Definition. The infinite series $a_1 + a_2 + \cdots + a_n + \cdots$, with partial sums $s_1, s_2, \ldots, s_n, \ldots$ is convergent if for any $\varepsilon > 0$, however small, there exists an N such that

$$|S - s_n| < \varepsilon \quad \text{for} \quad n > N \tag{1.2-3}$$

Let us explain this definition by means of an example.

Example 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \tag{1.2-4}$$

This series has the following partial sums:

$$\begin{aligned} s_1 &= a_1 = 1/2 \\ s_2 &= a_1 + a_2 = 2/3 \\ s_3 &= s_2 + a_3 = 3/4 \\ &\vdots \\ s_n &= n/n+1 \end{aligned}$$

Let us determine the limit of the partial sums

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n/n}{(n/n)+(1/n)} = 1. \tag{1.2-5}$$

The limit exists, therefore the series is convergent. Now let us apply the definition that is (1.2-3)

$$|S - s_n| = \left| 1 - \frac{n}{n+1} \right| = \left| \frac{n}{n+1} \right|$$

For $\varepsilon > 0$ there must be an N such that

$$\left| \frac{1}{n+1} \right| < \varepsilon \quad \text{for} \quad n > N$$

Let us take $\varepsilon = 0.1$, then what is N?

$$\frac{1}{n+1} < 0.1 \quad \therefore \quad 1 < 0.1(n+1)$$

and $1/0.1 = 10 < n+1 \quad \therefore \quad n > 9 \rightarrow N = 9$

Similarly for $\varepsilon = 1/100$ $N = 99$.

The remainder term in a series is defined as

$$r_n = S - s_n \tag{1.2-6}$$

Next we will give two examples where the series fails to converge.

Example 2.

$$\sum_{n=1}^{\infty} 1^n = 1 + 1 + 1 + \cdots$$

This series is clearly divergent since s_n increases without a limit as $n \rightarrow \infty$

Example 3.

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$$

This series has the following partial sums

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 0 \\
s_3 &= 1 \\
s_4 &= 0 \\
&\vdots
\end{aligned}
$$

The sequence of partial sums oscillates between 0 and 1 without reaching a limit, therefore this series is not convergent either.

The definition of convergence used in the examples above requires our knowledge of the partial sums. However, consider the following example.

Example 4.

$$0.1 + 0.01 + 0.001 + \cdots$$

with partial sums:

$$s_1 = 0.1; \; s_2 = 0.11; \; s_3 = 0.111 +$$

where each partial sum s_n being a decimal is less than 1, but on the other hand s_n increases as $n \to \infty$ and we cannot easily define the general form of s_n. In order to handle cases like the one above, we introduce the following theorem:

Theorem 1.2-1. If for an infinite series the partial sums s_n satisfy the condition

$$s_n \leqslant M \quad \text{and} \quad s_n \leqslant s_{n+1} \text{ for any } n$$

where M is some fixed number, then s_n has a limit and this limit is not greater than M. (This theorem is given without proof. In this book most theorems will be given without proofs, unless the proofs explain the theorems more fully. For a proof see Ref. 2.)

Next we will give a number of theorems concerning the convergence of series.

Theorem 1.2-2. If a series converges, then the general term a_n must approach zero as $n \to \infty$. If a_n does not approach zero as $n \to \infty$ the series diverges.

Proof: $s_n = s_{n-1} + a_n \quad \therefore \quad a_n = s_n - s_{n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0$$

Notice that the converse of this theorem is not true. That is if $a_n \to 0$ as $n \to \infty$, the series cannot be assumed to converge. We illustrate this by

Example 5.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

This is a well known divergent series, although $a_n \to 0 \; n \to \infty$.

Next we present the Cauchy convergence criterion, the most general theorem on convergence.

Theorem 1.2-3. If the series $\Sigma\, a_n$ converges, then for any $\varepsilon > 0$, there exists an N such that:

$$| a_m + a_{m+1} + \cdots + a_n | < \varepsilon \quad \text{for} \quad n > m \geqslant N$$

This theorem has the following converse: If for each $\varepsilon > 0$, there exists an N such that

$$| a_m + a_{m+1} + \cdots + a_n | < \varepsilon \quad \text{for} \quad n > m \geqslant N$$

then $\Sigma\, a_n$ converges.

Some properties of convergent series:

(1) A convergent series may be multiplied termwise by any constant

$$\Sigma \mathcal{P}\mathcal{A}_n = p\Sigma\, a_n = pS \tag{1.2-7}$$

(2) Two convergent series may be added term by term

$$\Sigma \tilde{a}_k + b_k) = \Sigma\, a_k + \Sigma\, b_k \tag{1.2-8}$$

(3) If a finite number of terms of an infinite convergent series are altered, the series' convergence is not affected. (Of course, the sum may change.)

1.2.2. Tests for Convergence

In this section we will present various tests for determining whether a series converges or diverges. However, before we get to the specific tests we introduce an integral known as the improper integral.

Definition. The Improper Integral: consider the following integral

$$\lim_{a,\, b \to \infty} \int_{-a}^{b} f(x)\, \mathrm{d}x = \lim_{a \to \infty} \int_{-a}^{0} f(x)\, \mathrm{d}x + \lim_{b \to \infty} \int_{0}^{b} f(x)\, \mathrm{d}x \tag{1.2-9}$$

If both of the limits in (1.2-9) exist, then

$$\int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = \lim_{r \to \infty} \int_{-r}^{+r} f(x)\, \mathrm{d}x \tag{1.2-10}$$

However, even when the two limits in (1.2-9) do not exist, it may be possible to assign a value to the expression given by (1.2-10). When this occurs, it is indicated by putting P in front of the integral in (1.2-10). The value so obtained is called the Cauchy principal value of the infinite integral and is written as:

$$P \int_{-\infty}^{+\infty} f(x)\, \mathrm{d}x = \lim_{r \to \infty} \int_{-r}^{+r} f(x)\, \mathrm{d}x \tag{1.2-11}$$

Example 1.

$$I = P\int_{-\infty}^{+\infty} x\,dx = \lim_{r\to\infty}\int_{-r}^{+r} x\,dx = \lim_{r\to\infty}\left[\frac{r^2}{2}-\frac{r^2}{2}\right] = 0 \tag{1.2-12}$$

It can be seen from the above example that, although each limit by itself is not bounded, the Cauchy principal value exists.

Example 2.

$$I = \int_{-1}^{2} \frac{1}{x}\,dx \tag{1.2-13}$$

The value of this integral does not exist in a strict sense as it blows up at $x = 0$, an interior point. To overcome this difficulty, consider

$$I = \int_{-1}^{2} \frac{dx}{x} = \lim_{\alpha\to 0}\int_{-1}^{0-\alpha} \frac{dx}{x} + \lim_{\beta\to 0}\int_{0+\beta}^{2} \frac{dx}{x} \tag{1.2-14}$$

$$I = \lim_{\alpha\to 0}\,[\log \alpha] + \lim_{\beta\to 0}\,[\log 2 - \log \beta] = \lim_{\alpha,\beta\to 0}\left[\log\left(\frac{\alpha}{\beta}\right) + \log 2\right] \tag{1.2-15}$$

Thus it is seen that I is multivalued, that is, the value it assumes depends on the ratio of α to β. We can achieve a unique result if we set $\alpha = \beta$. Then we obtain the Cauchy principal value.

$$P\int_{-1}^{2} \frac{dx}{x} = \lim_{\alpha\to 0}\left[\log\left(\frac{\alpha}{\alpha}\right) + \log 2\right] = \log 2 \tag{1.2-16}$$

Thus for a finite integral involving a function with a singularity at $x = \xi$ an interior point, the Cauchy principal value is defined as

$$P\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon\to 0}\left[\int_{a}^{\xi-\varepsilon} f(x)\,dx + \int_{\xi+\varepsilon}^{b} f(x)\,dx\right] \tag{1.2-17}$$

Example 3.

$$\int_{1}^{\infty} \frac{1}{x^{+p}}\,dx = \lim_{b\to\infty}\int_{1}^{b} x^{-p}\,dx = \lim_{b\to\infty}\frac{b^{1-p}-1}{1-p} \tag{1.2-18}$$

The convergence of (1.2-18) depends on the behavior of the b^{1-p} term. If $1 - p > 0$, then as $b \to \infty$ the integral diverges, however, if $1 - p < 0$ then $\lim_{b\to\infty} b^{1-p} = 1/b^{p-1} = 0$ and the integral converges as:

$$\int_{1}^{\infty} \frac{1}{x^{p}}\,dx = \frac{1}{p-1} \quad \text{for} \quad p > 1. \tag{1.2-19}$$

Theorem 1.2-4. (Cauchy Integral Test) For $x \geqslant 1$, let $f(x)$ be positive, continuous and decreasing, then the series

$$\sum_{n=1}^{\infty} f(n) \text{ and the integral } \int_1^{\infty} f(x)\,dx$$

both converge or diverge, and have the following partial sums

$$\int_1^{n+1} f(x)\,dx < \sum_{k=1}^{n} f(k) < \int_1^{n} f(x)\,dx + f(1) \tag{1.2-20}$$

Corollary 1. The improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if and only if $p > 1$.

Corollary 2. The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Let us illustrate these results by means of an example.

Example 1.

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{1+1^2} = \frac{1}{1+2^2} + \cdots + \frac{1}{1+n^2} + \cdots \tag{1.2-21}$$

is convergent and indicates its range. By the Cauchy Integral test (1.2-19) the series is convergent and is bounded by

$$\frac{\pi}{4} = \lim_{n\to\infty} \int_1^{n+1} \frac{1}{1+x^2}\,dx \leqslant \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leqslant \lim_{n\to\infty} \int_1^{n} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{1+1} = \frac{\pi}{4} + \frac{1}{2} \tag{1.2-22}$$

that is:

$$0.79 \leqslant \sum_{k=1}^{\infty} \frac{1}{1+k^2} \leqslant 1.29$$

Theorem 1.2-5. (Comparison Test). If $0 \leqslant a_n \leqslant b_n$, then the convergence of $\Sigma\, a_n$ follows from the convergence of $\Sigma\, b_n$. If $a_n \geqslant b_n \geqslant 0$, then the divergence of $\Sigma\, a_n$ follows from the divergence of $\Sigma\, b_n$.

Definition. a_n is said to be asymptotic to $b_n (a_n \sim b_n)$ if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \tag{1.2-23}$$

Corollary. If $a_n \sim b_n$ (are asymptotic), then the series $\Sigma\, a_n$ and $\Sigma\, b_n$ are both convergent or divergent.

Theorem 1.2-6. (The Ratio Test). The series $\Sigma\, a_n$ is convergent if:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$$

and divergent if

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1.$$

Notice that this test gives no information when the limit is equal to 1.

Example 2. Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \tag{1.2-24}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2}{2^{n+1}} \frac{2^n}{n^2}$$

$$= \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^2 \frac{1}{2} < 1$$

$\therefore$ the series converges.

Definition. A series $\Sigma\, a_n$ is said to be absolutely convergent if the series $\Sigma\, |a_n|$ is convergent.

Theorem 1.2-7. If a series is absolutely convergent then it is convergent.

Example 3.

$$\sum_{n=1}^{\infty} \frac{\cos nx}{2^n}$$

since $\left| \dfrac{\cos nx}{2^n} \right| = \dfrac{|\cos nx|}{2^n} \leqslant \dfrac{1}{2^n}$

and since . $1/2^n$ is a convergent series, the original series is absolutely convergent, thus convergent.

Theorem 1.2-8. The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

where $a_n > a_{n+1} > 0$ and $\lim_{n \to \infty} a_n = 0$ is convergent and the remainder term r_n is bounded by zero and the value of the first term not taken.

Definition. An alternating series as defined in theorem 1.2-8, which satisfies the above theorem but does not converge absolutely is said to be conditionally convergent.

Example 4. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots$$

does not converge absolutely, since $\Sigma\, 1/n$ is known to be divergent. But the alternating series satisfies the requirements of Theorem 1.2-8, since

$$a_n > a_{n+1} > 0 \rightarrow \frac{1}{n} > \frac{1}{n+1} > 0$$

and $\lim_{n\to\infty} 1/\mathrm{n}=0$, this series is conditionally convergent.

Properties of Absolutely Convergent Series

(1) The terms of an absolutely convergent series may be rearranged without changing its sum.
(2) Two absolutely convergent series may be multiplied like two finite series and their product rearranged in any manner.

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)=a_1b_1+(a_1b_2+a_2b_1)+(a_1b_3+a_2b_2+a_3b_1)+\cdots$$

1.2.3. Uniform Convergence

Consider a series which is a function of the variable x defined over a finite interval. The convergence of this series may or may not depend on the value of the variable x. Therefore we must introduce the following definition.

Definition. $\Sigma_{n=1}^{\infty} u_n(x)$ is said to be uniformly convergent in the interval $[\mathrm{a}, b]$ if for each $\varepsilon>0$, there exists an N, independent of x, such that $|S-s_n|<\varepsilon$ for all $n>N$.

Example 1. Consider the series

$$\sum_{n=0}^{\infty} x^n \text{ on } \left[-\frac{1}{2}, \frac{1}{2}\right].$$

$$S(x)=\frac{1}{1-x} \quad\text{and}\quad s_n(x)=\frac{1-x^n}{1-x}$$

note that these partial sums are not always known.

$$\therefore\quad S-s_n=\frac{x^n}{1-x}\quad\therefore\quad\left|\frac{x^n}{1-x}\right|<\varepsilon$$

and

$$n>\frac{\log\,\varepsilon(1-x)}{\log|x|}$$