WILEY FINANCE

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WILEY
A John Wiley & Sons, Ltd., Publication

This edition first published 2012

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

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Library of Congress Cataloging-in-Publication Data

Dynamic copula methods in finance / Umberto Cherubini . . . [et al.].
p. cm. — (The wiley finance series)
Includes bibliographical references and index.
ISBN 978-0-470-68307-1 (hardback)
1. Finance—Mathematical models. I. Cherubini, Umberto
HG106.D96 2011
332.01′519233—dc23
2011034154

A catalogue record for this book is available from the British Library.

ISBN 978-0-470-68307-1 (hardback) ISBN 978-1-119-95451-4 (ebk)
ISBN 978-1-119-95452-1 (ebk) ISBN 978-1-119-95453-8 (ebk)

Set in 10/12pt Times by Aptara Inc., New Delhi, India
Printed and bound by CPI Group (UK) Ltd, Croydon, CR0 4YY

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Preface

This book concludes five years of original research at the University of Bologna on the use of copulas in finance. We would like these results to be called the *Bologna school*. The problem tackled arises directly from financial applications and the fact that almost always in this field we are confronted with convolution problems along with non-normal distributions and non-linear dependence. More explicitly, almost always in finance we face the problem of evaluating the distribution of

$$X + Y$$

where X and Y may have arbitrary distributions and may be dependent on each other in quite a strange fashion. Very often, we may also be interested in the dependence of this sum on either X or Y. The *Bologna school* has studied the class of *convolution-based* copulas that is well suited to address this kind of problem. It is easy to see that this operates a restriction on the choice of copulas. In a sense, *convolution-based* copulas address a special *compatibility* problem, enforcing coherence in the dependence structure between variables and their sum. This compatibility issue is paramount and unavoidable for almost all the applications in finance. The first concept that comes to mind is the linear law of price enforced by the fundamental theorem of asset pricing: in order to avoid arbitrage, prices of complex products must be linear combinations of the primitive products constituting the replicating portfolio. In asset allocation, portfolios are also strictly linear concepts, even though they may include (and today they typically do) option-like and other non-linear products whose distribution is far from Gaussian and whose dependence on the other components of the portfolio is not Gaussian either. Moreover, trading and investment activities involve more and more exposures to credit risk that are non-Gaussian by definition: this was actually the very reason for copula function applications to finance in the first place. But, even in the case of credit, losses may be linked by the most complex dependence structure, but nevertheless they cumulate one after the other by a linear combination: computing cumulated losses is again a convolution problem. Finally, linear aggregation is crucial to understand the dynamics of markets. From this viewpoint, finance theory has developed under the main assumption of processes with independent increments: *convolution-based copulas* may allow us to considerably extend the set of possible market dynamics, allowing for general dependence structures between the price level at a given point in time and its increment (the return) in the following period: describing the distribution of the price at the end of a period is again a convolution problem.

The main message of this book is that copulas remain a flexible tool for applications in finance, but this flexibility is finite, and the *Bologna school* sets the frontier of this flexibility at the family of *convolution-based* copulas.

Chapters 1 and 2 review the general problem of dependence and correlation in finance. More particularly, Chapter 2 specializes the analysis to a review of the basic concepts of copulas, as they have been applied to financial problems until today. Chapters 3 and 4 introduce the theory of *convolution-based* copulas, and the concept of *C-convolution* within the mainstream of the Darsow, Nguyen, and Olsen (DNO) application of copulas to Markov processes. More specifically, Chapter 3 addresses theory and Chapter 4 deals with the application to econometrics. Chapters 5, 6, and 7 discuss applications of the approach in turn to the problems of: (i) evaluating multivariate equity derivatives; (ii) analyzing the credit risk exposure of a portfolio, (iii) aggregating Value-at-Risk measures across risk factors and business units. In all these chapters, we exploit the model to address dependence both in a spatial and temporal perspective. This twofold perspective is entirely new to these applications, and may easily be handled within the set of *convolution-based copulas*. Chapter 8 concludes by surveying other methodologies available in the mathematical finance and probability literature to set a dependence structure among processes: these approaches are mainly in continuous time, and raise the question, that we leave for future research, of whether they represent some or all the possible solutions that one would obtain by taking the continuous time limit of our model, which is defined in discrete time.

We conclude with thanks to our colleagues in the international community who have helped us during these years of work. Their support has been particularly precious, because our work is entirely free from government support. *Nemo propheta in patria*. As for comments on this manuscript, we would particularly like to thank, without implication, Xiaohong Chen, Fabrizio Durante, Marius Hofert, Matthias Scherer, Bruno Remillard, Paramsoothy Silvapulle, and an anonymous referee provided by John Wiley. And we thank our readers in advance for any comments they would like to share with us.

Contents

1
Correlation Risk in Finance

Over the last decade, financial markets have witnessed a progressive concentration of focus on correlation dynamics models. New terms such as *correlation trading* and *correlation products* have become the frontier topic of financial innovation. Correlation trading denotes the trading activity aimed at exploiting changes in correlation, or more generally in the dependence structure of assets and risk factors. Correlation products denote financial structures designed with the purpose of exploiting these changes. Likewise, the new term *correlation risk* in risk management is meant to identify the exposure to losses triggered by changes in correlation. Going long or short correlation has become a standard concept for everyone working in dealing rooms and risk management committees. This actually completes a trend that led the market to evolve from taking positions on the direction of prices towards taking exposures to volatility and higher moments of their distribution, and finally speculating and hedging on cross-moments. These trends were also accompanied by the development of new practices to transfer risk from one unit to others. In the aftermath of the recent crisis, these products have been blamed as one of the main causes. It is well beyond the scope of this book to digress on the economics of the crisis. We would only like to point out that the modular approach which has been typical of financial innovation in the *structured finance* era may turn out extremely useful to ensure the efficient allocation of risks among the agents. While on the one hand the use of these techniques without adequate knowledge may represent a source risk, avoiding them for sure represents a distortion and a source of cost. Of course, accomplishing this requires the use of modular mathematical models to split and transfer risk. This book is devoted to such models, which in the framework of dependence are called *dependence functions* or *copula functions*.

1.1 CORRELATION RISK IN PRICING AND RISK MANAGEMENT

In order to measure the distance between the current practice of markets and standard textbook theory of finance, let us consider the standard static portfolio allocation problem. The aim is to maximize the expected utility of wealth W at some final date T using a set of risky assets, $S_i, i = 1, \ldots, m$. Formally, we have

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{U}\left(R_f + \sum_{i=1}^{m} w_i(R_i - R_f)\right)\right],$$

where $R_i = \ln(S_i(T)/S_i(0))$ are the log-returns on the risky assets and R_f is the risk-free rate. The asset allocation problem is completely described by two functions: (i) the utility function $\mathbb{U}(.)$, assumed strictly increasing and concave; (ii) the joint distribution function of the returns $\mathbb{P}$. While we could argue in depth about both of them, throughout this book the focus will be on the specification of the joint distribution function. In the standard textbook problem, this is actually kept in the background and returns are assumed to be jointly normally distributed, which leads to rewriting the expected utility in terms of a mean–variance problem.

Nowadays, real-world asset management has moved miles away from this textbook problem, mainly for two reasons: first, investments are no longer restricted to linear products, such as stocks and bonds, but involve options and complex derivatives; second, the assumption that the distribution of returns is Gaussian is clearly rejected by the data. As a result, the expected utility problem should take into account three different dimensions of risk: (i) directional movements of the market; (ii) changes in volatility of the assets; (iii) changes in their correlation. More importantly, there is also clear evidence that changes in both volatility and correlation are themselves correlated with swings in the market. Typically, both volatility and correlation increase when the market is heading downward (which is called the *leverage effect*). It is the need to account for these new dimensions of risk that has led to the diffusion of derivative products to hedge against and take exposures to both changes in volatility and changes in correlation. In the same setting, it is easy to recover the other face of the same problem encountered by the pricer. From his point of view, the problem is tackled from the first-order conditions of the investment problem:

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{U}'\left(R_f+\sum_{i=1}^{m}w_i(R_i-R_f)\right)(R_i-R_f)\right]=0=\mathbb{E}_{\mathbb{Q}}\left[R_i-R_f\right],$$

where the new probability measure $\mathbb{Q}$ is defined after the Radon–Nikodym derivative

$$\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}=\frac{\mathbb{U}'\left(R_f+\sum_{i=1}^{m}w_i(R_i-R_f)\right)}{\mathbb{E}_{\mathbb{P}}\left[\mathbb{U}'\left(R_f+\sum_{i=1}^{m}w_i(R_i-R_f)\right)\right]}.$$

Pricers face the problem of evaluating financial products using measure $\mathbb{Q}$, which is called the *risk-neutral measure* (because all the risky assets are expected to yield the same return as the risk-free asset), or the *equivalent martingale measure* (EMM, because $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$ with the property that prices expressed using the risk-free asset as *numeraire* are martingale). An open issue is whether and under what circumstances volatility and correlation of the original measure $\mathbb{P}$ are preserved under this change of measure. If this is not the case, we say that volatility and correlation risks are priced in the market (that is, a risk premium is required for facing these risks). Under this new measure, the pricers face problems which are similar to those of the asset manager, that is evaluating the sensitivity of financial products to changes in the direction of the market (long/short the asset), volatility (long/short volatility) and correlation (long/short correlation). They face a further problem, though, that is going to be the main motivation of this book: they must ensure that prices of multivariate products are consistent with prices of univariate products. This consistency is part of the so-called *arbitrage-free* approach to pricing, which leads to the martingale requirement presented above. In the jargon of statisticians, this consistency leads to the term *compatibility*: the risk-neutral joint distribution $\mathbb{Q}$ has to be compatible with the marginal distributions $\mathbb{Q}_i$.

Like the asset manager and the pricer, the risk manager also faces an intrinsically multivariate problem. This is the issue of measuring the exposure of the position to different risk factors. In standard practice, he transforms the financial positions in the different assets and markets into a set of exposures (*buckets*, in the jargon) to a set of risk factors (*mapping process*). The problem is then to estimate the joint distribution of losses on these exposures and define a risk measure on this distribution. Typical measures are *Value-at-Risk* (VaR) and *Expected Shortfall* (ES). These measures are multivariate in the sense that they must account for correlation among the losses, but there is a subtle point to be noticed here, which makes this practice obsolete with respect to structured finance products, and correlation products in particular.

A first point is that these products are non-linear, so that their value may change even though market prices do not move but their volatilities do. As for volatility, the problem can be handled by including a *bucket* of volatility exposures for every risk factor. But there is a crucial point that gets lost if correlation products are taken into account. It is the fact that the value of these products can change even if neither the market prices nor their volatilities move, but simply because of a change in correlation. In fact, this exposure to correlation among the assets included in the specific product is lost in the mapping procedure. Correlation risk then induces risk managers to measure this dimension of risk on a product-by-product basis, using either historical simulation or stress-testing techniques.

1.2 IMPLIED VS REALIZED CORRELATION

A peculiar feature of applications of probability and statistics to finance is the distinction between historical and implied information. This duality, that is found in many (if not all) applications in univariate analysis, shows up in the multivariate setting as well. On the one side, standard time series data from the market enable us to gauge the relevance of market co-movements for investment strategies and risk management issues. On the other side, if there exist derivative prices which are dependent on market correlation, it is possible to recover the degree of co-movement credited by investors and financial intermediaries to the markets, and this is done by simply inverting the prices of these derivatives. Of course, recovering implied information is subject to the same flaws as those that are typical of the univariate setting. First, the possibility of neatly backing out this information may be limited by the market incompleteness problem, which has the effect of introducing a source of noise into market prices. Second, the distribution backed out is the risk-neutral one and a market price of risk could be charged to allow for the possibility of correlation changes. These problems are indeed compounded and in a sense magnified in the multivariate setting, in which the uncertainty concerning the dependence structure among the markets adds to that on the shape of marginal distributions.

Unfortunately, there are not many cases in which correlation can be implied from the market. An important exception is found in the FOREX market, because of the so-called *triangular arbitrage* relationship. Consider the Dollar/Euro ($e_{US,E}$), the Euro/Yen ($e_{E,Y}$) and the Dollar/Yen ($e_{US,Y}$) exchange rates. Triangular arbitrage requires that

$$e_{US,E} = e_{E,Y} e_{US,Y}.$$

Taking logs and denoting by $\sigma_{US,E}$, $\sigma_{E,Y}$, and $\sigma_{US,Y}$ the corresponding implied volatilities, we have that

$$\sigma^2_{US,E} = \sigma^2_{E,Y} + \sigma^2_{US,Y} + 2\rho\sigma_{E,Y}\sigma_{US,Y},$$

from which

$$\rho = \frac{\sigma^2_{US,E} - \sigma^2_{E,Y} - \sigma^2_{US,Y}}{2\sigma_{E,Y}\sigma_{US,Y}}$$

is the implied correlation between the Euro/Yen and the Dollar/Yen priced by the market.

1.3 BOTTOM-UP VS TOP-DOWN MODELS

For all the reasons above, estimating correlation, either historical or implied, has become the focus of research in the last decade. More precisely, the focus has been on the specification of the joint distribution of prices and risk factors. This has raised a first strategic choice between two opposite classes of models, that have been denoted *top-down* and *bottom-up* approaches. In all applications, pricing of equity and credit derivatives, risk management aggregation and allocation, the first choice is then to fit all markets and risk factors with a joint distribution and to specify in the process both the marginal distributions of the risk factors and their dependence structure. The alternative is to take care of marginal distributions first, and of the dependence structure in a second step. It is clear that copula functions represent the main tool of the latter approach. It is not difficult to gauge what the pros and cons of the two alternatives might be. Selecting a joint distribution fitting all risks may not be easy, beyond the standard choices of the normal distribution for continuous variables and the Poisson distribution for discrete random variables. If one settles instead for the choice of non-parametric statistics, even for a moderate number of risk factors, the implementation runs into the so-called *curse of dimensionality*. As for the advantages, a top-down model would make it fairly easy to impose restrictions that make prices consistent with the equilibrium or no-arbitrage restrictions. Nevertheless, this may come at the cost of marginal distributions that do not fit those observed in the market. Only seldom (to say never) does this poor fit correspond to arbitrage opportunities, while more often it is merely a symptom of *model risk*. On the opposite side, the bottom-up model may ensure that marginal distributions are properly fitted, but it may be the case that this fit does not abide by the consistency relationships that must exist among prices: the most well known example is the no-arbitrage restriction requiring that prices of assets in speculative markets follow martingale processes. The main goal of this book is actually to show how to impound restrictions like these in a bottom-up framework.

1.4 COPULA FUNCTIONS

Copula functions are the main tool for a bottom-up approach. They are actually built on purpose with the goal of pegging a multivariate structure to prescribed marginal distributions. This problem was first addressed and solved by Abe Sklar in 1959. His theorem showed that any joint distribution can be written as a function of marginal distributions:

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

and that the class of functions $C(.)$, denoted copula functions, may be used to extend the class of multivariate distributions well beyond those known and usually applied. To quote the dual approach above, the former result allows us to say that any top-down approach may be written in the formalism of copula functions, while the latter states that copulas can be applied in a bottom-up approach to generate infinitely many distributions. A question is whether this multiplicity may be excessive for financial applications, and this whole book is devoted to that question.

Often a more radical question is raised, whether there is any advantage at all to working with copulas. More explicitly, one could ask what can be done with copulas that cannot be done with other techniques. The answer is again the essence of the bottom-up philosophy. The crucial point is that in the market, we are used to observing marginal distributions. All the information that we can collect is about marginals: the time series of this and that price, and the implied distribution of the underlying asset of an option market for a given exercise date. We can couple time series of prices or of distributions together and study their dependence, but only seldom can we observe multivariate distributions. For this reason, it is mandatory that any model be made consistent with the univariate distributions observed in the market: this is nothing but an instance of that procedure pervasively used in the markets and called *calibration*.

1.5 SPATIAL AND TEMPORAL DEPENDENCE

To summarize the arguments of the previous section, it is of the utmost importance that multivariate models be consistent with univariate observed prices, but this consistency must be subject to some rules and cannot be set without limits. These limits were not considered in standard copula functions applications to finance problems. In these applications the term *multivariate* was used with the meaning that several different risk factors at a given point in time were responsible for the value of a position at that time. This concept is called *spatial dependence* in statistics and is also known as *cross-section* dependence in econometrics. Copula functions could be used in full flexibility to represent the consistency between the price of a multivariate product at a given date and the prices of the constituent products observed in the market. However, the term multivariate could be used with a different meaning, that would make things less easy. It could in fact refer to the dependence structure of the value of the same variable observed at different points in time: this is actually defined as a *stochastic process*. In the language of statistics, the dependence among these variables would be called *temporal dependence*. Curiously, in econometric applications copula functions have mainly been intended in this sense. If copulas are used in the same sense in derivative pricing problems, the flexibility of copulas immediately becomes a problem: for example, one would like to impose restrictions on the dynamics to have Markov processes and martingales, and only a proper specification of copulas could be selected to satisfy these requirements.

Even more restrictions would apply in an even more general setting, in which a multivariate process would be considered as a collection of random variables representing the value of each asset or risk factor at different points in time. In the standard practice of econometrics, in which it is often assumed that relationships are linear, this would give rise to the models called *vector autoregression* (VAR). Copula functions allow us to extend these models to a general setting in which relationships are allowed to be non-linear and non-Gaussian, which is the rule rather than the exception of portfolios of derivative products.

1.6 LONG-RANGE DEPENDENCE

These models that extend the traditional VAR time series with a specification in terms of copula functions are called *semi-parametric* and the most well known example is given by the so-called SCOMDY model (Semiparametric Copula-based Multivariate Dynamic). By taking the comparison with linear models one step further, these models raise the question of the behavior of the processes over long time horizons. We know from the theory of time series that a univariate process y_t modeled with the dynamics

$$y_t = \omega + \epsilon_t + \alpha_1 \epsilon_{t-1} + \ldots + \alpha_q \epsilon_{t-q} + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p},$$

where ω, $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ are constant parameters, $\epsilon_{t-i} \sim N(0, \sigma)$ and $Cov(\epsilon_{t-i}, \epsilon_{t-j}) = 0, i \neq j$ is called an ARMA(p,q) model (Autoregressive Moving Average

Process), that could be extended to a multiple set of processes called VARMA. In particular, the MA part of the process is represented by the dependence of y_t on the past q innovations ϵ_{t-i} and the AR part is given by its dependence on the past p values of the process itself. If we focus on the autoregressive part, we know that in cases in which the characteristic equation of the process

$$z - \beta_1 z - \ldots - \beta_p z^p = 0$$

has solutions strictly inside the unit circle, the process is said to be *stationary* in mean. To make the meaning clear, let us just focus on the simplest AR(1) process:

$$y_t = \omega + \epsilon_t + \beta_1 y_{t-1}.$$

It is easy to show that if $\beta_1 < 1$ by recursive substitution of y_{t-i-1} into y_{t-i}, we have

$$\mathbb{E}(y_t) = \omega \sum_{i=0}^{\infty} \beta_1^i + \sum_{i=0}^{\infty} \beta_1^i \mathbb{E}(\epsilon_{t-i}) = \frac{\omega}{1-\beta_1}$$

and

$$VAR(y_t) = \sum_{i=0}^{\infty} \mathbb{E}(\beta_1^i \epsilon_{t-i})^2 = \frac{\sigma^2}{1-\beta_1^2},$$

where we have used the moments of the distribution of ϵ_{t-i}. Notice that if instead it is $\beta_1 = 1$, the dynamics of y_t is defined by

$$y_t = \omega + \epsilon_t + y_{t-1}$$

and neither the mean nor the variance of the unconditional distribution are defined. In this case the process is called *integrated* (of order 1) or *difference stationary*, or we say that the process contains a *unit root*. The idea is that the first difference of the process is stationary (in mean). The distinguishing feature of these processes is that any shock affecting a variable remains in its history forever, a property called *persistence*. As an extension, one can conceive that several processes may be linear combinations of the same persistent shock y_t, that is also called the *common stochastic trend* of the processes. In this case we say that the set of processes constitutes a *co-integrated* system. More formally, a set of processes is said to constitute a co-integrated system if there exists at least one linear combination of the processes that is stationary in mean.

In another stream of literature, another intermediate case has been analyzed, in which the process is said to be *fractionally* integrated, so that the process is made stationary by taking fractional differences: the long-run behavior of these processes is denoted *long memory*. In Chapter 4 we shall give a formal definition of long memory (due to Granger, 2003) and we will discuss the linkage with a copula-based stochastic process. As for the contribution of copulas to these issues, notice that while most of the literature on unit roots and persistence vs stationary models has developed under the maintained assumption of Gaussian innovations, the use of copula functions extends the analysis to non-Gaussian models. Whether these models can represent a new specification for the long-run behavior of time series remains an open issue.